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Informationally optimal correlation

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Abstract This papers studies an optimization problem under entropy constraints arising from repeated games with signals. We provide general properties of solutions and a full characterization of optimal solutions for 2×2 sets of actions. As an application we compute the min max values of some repeated games with signals.

Keywords Correlation · Entropy · Repeated games

Mathematics Subject Classification (2000) 90C26 · 94A15 · 91A20

1 Introduction

A probability distribution D on a product set $A = \prod_{i \in N} A^i$ can be represented as a convex combination of independent distributions $D = \sum_{k=1}^{K} \alpha_k d_k$ in a variety of

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ways. This paper looks into the problem of finding the decomposition $\left\{(\alpha_k^*, d_k^*)\right\}_{k=1}^{K}$ of a distribution D with maximal expected entropy:

$$\left\{(\alpha_k^*, d_k^*)\right\}_{k=1}^{K} \in \arg \max_{\sum_{k=1}^{K} \alpha_k d_k = D} \sum_{k=1}^{K} \alpha_k H(d_k),$$

where the d_k's are independent probability distribution on A and H is the entropy function.

The motivation of this work stems from the computation of individually rational levels in repeated games with imperfect monitoring, which itself comes from the computation of Nash equilibrium payoffs in such repeated games. The celebrated Folk theorem due to Aumann and Shapley [1] asserts that in repeated games with long horizon and perfect monitoring of actions (when each player gets to observe at each stage the actions chosen by all players during the previous stage), Nash equilibrium payoffs coincide with feasible and individually rational payoffs vectors. A vector payoff is called feasible if it can be induced by some strategy profile. It is individually rational when for every player, it is superior to his min max payoff, defined as the minimum to which other players can force this player down. The main rationale behind this result is that players agree on a rule to select the sequence of action profiles and, whenever players others than i see player i "cheating" from the prescribed rule, they "punish" player i by using min max strategies against him in the repeated game. These punishment threats are sufficient to deter any player from cheating when the payoff implemented by the prescribed rule is individually rational.

A central open problem in the theory of repeated games is the extension of the "Folk Theorem" to repeated games with imperfect monitoring, in which each player gets to observe at each stage a (partially informative) signal on the actions chosen during the previous stage. Since, under imperfect monitoring as well as under perfect monitoring, equilibrium payoffs are feasible and individually rational, the computation of min max payoffs is an essential step towards a characterization of equilibrium payoffs.

In repeated games with perfect monitoring, the min max level for player i is the min max of the stage game given by the formula $\min_{s^{-i}} \max_{a^i} g^i(s^{-i}, a^i)$, where s^{-i} is a profile of (independent) mixed strategies of other players than i, a^i is player i's action, and g^i is i's stage payoff function.

In repeated games with imperfect monitoring, information asymmetries about past play may create possibilities of correlation for a group of players.

For instance, if all players except i have perfect monitoring and if player i observes no signals, player i's opponents can exchange messages that are secret for player i and punish him to the min max level in correlated mixed strategies, given by $\min_{d^{-i}} \max_{a^i} g^i(s^{-i}, a^i)$, where d^{-i} is any (possibly correlated) distribution of actions of other players but i.

In general games with imperfect monitoring, the min max level for a player lies between the correlated min max and the min max in mied strategies of the one-shot game.

The characterization of min max payoffs of general repeated games with imperfect monitoring is an open problem. This paper solves the question for some classes signalling structures. It develops some tools and shows potential directions of investigation for more general signalling structures.

Our method relies on Gossner and Tomala [5] who study the difference of forecasting abilities between a perfect observer of a stochastic process and an observer who gets imperfect signals on the same process. Building on this result, Gossner and Tomala [6] consider repeated games where player i gets a signal on his opponents' action profile which does not depend on his own action. At a given stage of the game, i holds a belief on the mixed action profile used by players against him, represented by a probability distribution on the set of uncorrelated mixed action profiles. Such a distribution, Z, is called a *correlation system*.

To each correlation system corresponds an entropy variation, $\Delta H(Z)$, defined as the difference between the expected entropy of the mixed action profile of players against i and the entropy of the signal observed by i. Gossner and Tomala [6] prove that the max min of the repeated game (where player i is minimizing) is the highest payoff obtained by using two correlation systems Z and Z' with respective time frequencies $\lambda, 1-\lambda$ under the constraint that the average entropy variation is non-negative (*i.e.* $\lambda \Delta H(Z) + (1-\lambda)\Delta H(Z') \geq 0$). To achieve this payoff, the opponents of i start by generating signals that give little information to player i (they accumulate entropy). Then they play alternatively a correlation system that yields a bad payoff but generates entropy (has a positive entropy variation) and another that uses the entropy just generated to yield a good payoff. The constraint on the frequencies of the correlation system is that on average, the entropy variation must be greater than or equal to zero.

The aim of the present paper is to develop tools for computing optimal solutions of this problem when the team against player i consists of two players. Fixing a correlated distribution of actions, we select, among the correlation systems that induce it, the one with maximal expected entropy. We derive general properties of the solutions and a full characterization of these solutions when each of the team player's action spaces has two elements. Relying on these solutions, we deduce a full analytic characterization of the max min of an example of repeated game with imperfect monitoring. Another application of our characterization of optimal correlation systems has been developed by Goldberg [4]. Beyond the game studied in this paper, the tools we develop may serve as a basis for computations of solutions of maximization problems under entropy constraints raising from other optimization or game theoretic problems.

This paper is part of a growing body of literature on entropy methods in repeated games. Lehrer and Smorodinsky [9] relate the relative entropy of a probability measure P with respect to a belief Q and the merging of P to Q. Neyman and Okada [11,12] use entropy as a measure of the randomness of a mixed strategy, and apply it to repeated games played by players with bounded rationality. Gossner and Vieille [7] compute the max min value of a zero-sum repeated game where the maximizing player is not allowed to randomize freely but privately observes an exogenous i.i.d. process, and show that this value depends on the exogenous process through its entropy only. Gossner et al. [3] apply entropy methods to the study of optimal use of communication resources.

We present the notion of informationally optimal correlation system and our main results in Sect. 2. Section 3 presents the application to repeated game problems. The main proofs are in Sect. 4.

2 Informationally optimal correlation

2.1 Model and definitions

Let $N = \{1, \dots, n\}$ be a finite team of players and A^i be a finite set of actions for player $i \in N$. A mixed strategy for player i is a probability distribution x^i on A^i and we let $X^i = \Delta(A^i)$ be the set of probability distributions on A^i. We let $A = \prod_{i \in N} A^i$ be the set of action profiles and $X^N = \Delta(A)$ be the set of (correlated) probability distributions on A. We also let $X = \otimes_{i \in N} X^i$ the set of independent probability distributions on A, i.e. a distribution D is in X if there exist $x^1 \in X^1, \dots, x^n \in X^n$ such that for each a, $D(a) = \prod_i x^i(a^i)$, we write then $D = \otimes_i x_i \in \Delta(A)$.

We describe how correlation of actions is obtained. A finite random variable $\mathbf{k}$ with law $p = (p_k)_k$ is drawn and announced to each player in the team and to no one else. Then each player chooses an action, possibly at random. We think of $\mathbf{k}$ as a common information shared by the team's members which is secret for an external observer. For example, $\mathbf{k}$ can be the result of secret communication within the team, or it can be provided by a correlation device (Aumann, [2]). Conditioning the mixed strategies on the value of $\mathbf{k}$, the team can generate every distribution of actions of the form

$$D = \sum_k p_k \otimes_i x_k^i$$

for each k, $x_k^i \in X^i$. The distribution D can thus be seen as the belief of the external observer on the action profile played by the team. Note that the random variable $\mathbf{k}$ intervenes in the decomposition through its law only and in fact only through the distribution it induces on mixed strategies. We define thus a correlation system as follows

Definition 1 A *correlation system* Z is a distribution with finite support on X:

$$Z = \sum_{k=1}^{K} p_k \epsilon_{\otimes_i x_k^i}$$

where for each k, $p_k \geq 0$, $\sum_k p_k = 1$, for each i, $x_k^i \in X^i$, and $\epsilon_{\otimes_i x_k^i}$ stands for the Dirac measure on $\otimes_i x_k^i$.

The *distribution of actions* induced by Z is $D(Z) = \sum_k p_k \otimes_i x_k^i$, element of X^N.

We measure the randomness of correlation systems using the information theoretic notion of entropy.

Let $\mathbf{x}$ be a finite random variable with law p, the entropy of $\mathbf{x}$ is: $H(\mathbf{x}) = \mathbf{E}[-\log p(\mathbf{x})] = -\sum_x p(x) \log p(x)$, where $0 \log 0 = 0$ and the logarithm is in basis 2. $H(\mathbf{x})$ is non-negative and depends only on p, we shall thus also denote it $H(p)$. Let $(\mathbf{x}, \mathbf{y})$ be a pair of finite random variables with joint law p. For each x, y, define the conditional entropy of $\mathbf{x}$ given $\mathbf{y}$ by: $H(\mathbf{x}|\mathbf{y}) = -\sum_{x,y} p(x, y) \log p(x|y)$. Entropy verifies the following chain rule: $H(\mathbf{x}, \mathbf{y}) = H(\mathbf{y}) + H(\mathbf{x}|\mathbf{y})$. In the case of a binary distribution $(p, 1 - p)$ we let,

$$h(p) := H(p, 1-p) = -p \log p - (1-p) \log(1-p)$$

The uncertainty of an observer regarding the action profile of the team is the result of two effects: (1) team players condition their actions on the random variable $\mathbf{k}$, (2) conditional on the value of $\mathbf{k}$ team players use mixed actions x_k^i. We measure the uncertainty generated by the team itself by the expected entropy of $\otimes_i x_k^i$.

Definition 2 Let Z be a correlation system, $Z = \sum_{k=1}^K p_k \epsilon_{\otimes_i x_k^i}$. The *expected entropy* of Z is

$$J(Z) = \sum_k p_k H(\otimes_i x_k^i)$$

Example 1 Consider a two-player team, with two actions for each player: $A^1 = A^2 = \{G, H\}$. We identify a mixed strategy for player i with the probability it puts on G. A distribution $D \in X^{12}$ is denoted $D = \begin{pmatrix} d_1 & d_2 \\ d_3 & d_4 \end{pmatrix}$, where d_1 denotes the probability of the team's action profile (G, G), d_2 the probability of (G, H), etc. The distribution $D = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$ can be uniquely decomposed as a convex combination of independent distributions as follows: $D = \frac{1}{2}(1 \otimes 1) + \frac{1}{2}(0 \otimes 0)$. A correlation system Z such that $D(Z) = D$ is thus uniquely defined: $Z = \frac{1}{2}\epsilon_{1\otimes 1} + \frac{1}{2}\epsilon_{0\otimes 0}$, i.e. the players flip a fair coin an play (G, G) if heads and (H, H) if tails. Then given $\mathbf{k} = k$, the strategies used are pure, thus $J(Z) = \sum_k p_k H(\otimes_i x_k^i) = 0$.

By contrast the distribution $D' = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} \end{pmatrix}$ can be obtained by several correlation system. For example, $D' = D(Z)$ for the following Z's:

- $Z_1 = \frac{1}{3}\epsilon_{1\otimes 1} + \frac{1}{3}\epsilon_{1\otimes 0} + \frac{1}{3}\epsilon_{0\otimes 0}$.
- $Z_2 = \frac{2}{3}\epsilon_{1\otimes \frac{1}{2}} + \frac{1}{3}\epsilon_{0\otimes 0}$.
- $Z_3 = \frac{1}{2}\epsilon_{1\otimes \frac{2}{3}} + \frac{1}{2}\epsilon_{\frac{1}{3}\otimes 0}$

Under Z_1, the players play pure strategies conditional on the value of k, thus $J(Z_1) = 0$. Under Z_2, $J(Z_2) = \sum_k p_k H(\otimes_i x_k^i) = \frac{2}{3} H(\frac{1}{2}, \frac{1}{2}) = \frac{2}{3}$. Under Z_3, $\sum_k p_k H(\otimes_i x_k^i) = H(\frac{1}{3}, \frac{2}{3})$. One gets then $J(Z_3) > J(Z_2) > J(Z_1)$. The question is how to generate D' with maximal expected entropy? It turns out that Z_3 is optimal for D' in this sense. This leads to the following definition.

Definition 3 Given $D \in X^N$, a correlation system Z is *informationally optimal for* D if:

1. $D(Z) = D$;
2. For every Z' such that $D(Z') = D$, $J(Z') \leq J(Z)$.

In other words, Z is a solution of the optimization problem:

$$\max_{Z: D(Z) = D} J(Z) \quad (P_D)$$

A correlation system Z is *informationally optimal* if it is informationally optimal for $D(Z)$.

2.2 Properties

Now we prove the existence of optimal correlation systems for every distribution D.

Proposition 1 *For every $D \in X^N$, there exists Z optimal for D which has finite support of cardinal no more than $\prod_i \left|A^i\right| + 1$.*

Proof Let $D \in X^N$, identifying an action a^i of player i with the mixed strategy $\epsilon_{a^i} \in X^i$, one has

$$D = \sum_a D(a) \otimes_i a^i$$

Thus the set of Z such that $D(Z) = D$ is non-empty. Now for each $Z = \sum_{k=1}^K p_k \epsilon_{\otimes_i x_k^i}$ such that $D(Z) = D$, the vector $(D(Z), J(Z))$ writes

$$(D(Z), J(Z)) = \sum_{k=1}^K p_k \left(\otimes_i x_k^i, H(\otimes_i x_k^i) \right)$$

and thus belongs to the convex hull of the set

$$S = \left\{ (\otimes_i x^i, H(\otimes_i x^i)) \mid \otimes_i x^i \in X \right\}$$

S is a subset of $\Delta(A) \times \mathbb{R}$ which has dimension $(\prod_i \left|A^i\right| - 1) + 1$. From Carathéodory's theorem, $(D(Z), J(Z))$ can be obtained by a convex combination of at most $K = \prod_i \left|A^i\right| + 1$ points in S. Summing up, for each distribution D and correlation system Z s.t. $D(Z) = D$, there exists Z' with $\left|\text{supp } Z'\right| \leq K$, $D(Z') = D$ and $J(Z') = J(Z)$. It is plain that the set of correlation systems Z' s.t. $\left|\text{supp } Z'\right| \leq K$ and $D(Z') = D$ is a nonempty finite dimensional compact set and that the mapping J is continuous on it. The maximum of J is thus attained on this set. □

Solutions to the problem (P_D): $\max_{Z:D(Z)=D} J(Z)$ thus exist. We establish some properties on the value of (P_D).

Proposition 2
1. *The mapping $\varphi\colon D \mapsto$ value of P_D is the smallest concave function on X^N such that its restriction to X, $\varphi_{|X}$ is pointwise (weakly) greater than the entropy function,* i.e. $\varphi(\otimes_i x^i) \geq H(\otimes_i x^i)$ *for each $\otimes_i x^i \in X$.*
2. *φ is continuous on X^N.*
3. *For each D, $\varphi(D) \leq H(D)$. Furthermore, $\varphi(D) = H(D)$ iff D is a product distribution.*

Proof (1) Let f be the bounded mapping $f : X^N \to \mathbb{R}$, such that

$$f(D) = \begin{cases} H(D) & \text{if } D \in X \\ 0 & \text{if } D \notin X \end{cases}$$

Then $\varphi = \operatorname{cav} f$ the smallest concave function on X^N that is pointwise (weakly) greater than f.

(2) Since f is uppersemicontinuous and X^N is a polytope, we deduce from Laraki [8] (theorem 1.16, proposition 2.1 and proposition 5.2) that φ is uppersemicontinuous. Also, since X^N is a polytope and φ is bounded and concave, we deduce from Rockafellar [13] (theorem 10.2 and theorem 20.5) that φ is lowersemicontinuous.

(3) If $D = \sum_k p_k \otimes_i x_k^i$, by concavity of the entropy function, $H(D) \geq \sum_k p_k H(\otimes_i x_k^i)$, thus $H(D) \geq \varphi(D)$. Assume $D \in X$, i.e. $D = \otimes_i x^i$, by point (1) $\varphi(\otimes_i x^i) \geq H(\otimes_i x^i)$ so that $\varphi(\otimes_i x^i) = H(\otimes_i x^i)$. If $D \notin X$, from proposition 1 there exists $Z = \sum_{k=1}^K p_k \epsilon_{\otimes_i x_k^i}$ s.t. $D = \sum_k p_k \otimes_i x_k^i$ and $\varphi(D) = \sum_k p_k H(\otimes_i x_k^i)$ and by strict concavity of the entropy function, $\varphi(D) < H(D)$. □

The set of optimal correlation systems possesses a kind of consistency property. Roughly, one cannot find in the support of an optimal system, a sub-system which is not optimal. In geometric terms, if we denote by $\mathcal{Z}$ the set of all correlation systems and $\mathcal{F}(Z)$ the minimal geometric face of the convex $\mathcal{Z}$ containing Z, then the following lemma states that if Z is optimal then any correlation system that belongs to $\mathcal{F}(Z)$ is also optimal (for a precise definition of the geometric face in infinite dimension, see e.g. [8]).

Lemma 1 *If Z is informationally optimal and* $\operatorname{supp} Z' \subseteq \operatorname{supp} Z$ *then Z' is also informationally optimal.*

In particular, if $Z = \sum_{k=1}^K p_k \epsilon_{\otimes_i x_k^i}$ is informationally optimal, then for any k_1 and k_2 in $\{1, \ldots, K\}$ such that $p_{k_1} + p_{k_2} > 0$, $\frac{p_{k_1}}{p_{k_1}+p_{k_2}} \epsilon_{\otimes_i x_{k_1}^i} + \frac{p_{k_2}}{p_{k_1}+p_{k_2}} \epsilon_{\otimes_i x_{k_2}^i}$ is informationally optimal.

Proof The set of Z' such that $\operatorname{supp} Z' \subseteq \operatorname{supp} Z$ is the minimal face containing Z of the set of correlations systems. Therefore, Z lies in the relative interior of this face (from the previous lemma, we can bound uniformly the support and assume all Z's to be in some finite dimensional space). So for each Z' s.t. $\operatorname{supp} Z' \subseteq \operatorname{supp} Z$, there exist $0 < \lambda \leq 1$ and Z'' such that $Z = \lambda Z' + (1 - \lambda) Z''$. Assuming that Z' is not informationally optimal, there exists Z^* s.t. $D(Z^*) = D(Z')$ and $J(Z^*) > J(Z')$. Define $Z^0 = \lambda Z^* + (1 - \lambda) Z''$, then $D(Z^0) = D(Z)$ and $J(Z^0) - J(Z) = \lambda(J(Z^*) - J(Z'))$ contradicting the optimality of Z. □

2.3 Characterization in the 2 × 2 case

We characterize informationally optimal correlation systems for two player teams where each team player possesses two actions. We assume from now on that $A^1 = A^2 = \{G, H\}$. We identify a mixed strategy x (resp. y) of player 1 (resp. 2) with the probability of playing G, i.e. to a number in the interval [0, 1]. We denote distributions $D \in X^{12}$ by

$$D = \begin{pmatrix} d_1 & d_2 \\ d_3 & d_4 \end{pmatrix},$$

where d_1 denotes the probability of the team's action profile (G, G), d_2 the probability of (G, H), etc.

The following theorem shows that the informationally optimal correlation system associated to any D is unique, contains at most two elements in its support, can be easily computed for a given distribution, and that the set of informationally optimal correlation systems admits a simple parametrization.

Theorem 1 *For every $D \in X^{12}$, there exists a unique Z^D which is informationally optimal for D. Moreover,*

– *If $\det(D) = 0$, $Z^D = \epsilon_{x\otimes y}$ where:*

$$x = d_1 + d_2, \;\; y = d_1 + d_3$$

– *If $\det(D) < 0$, $Z^D = p\epsilon_{x\otimes y} + (1-p)\epsilon_{y\otimes x}$ where x and y are the two solutions of the second degree polynomial equation*

$$X^2 - (2d_1 + d_2 + d_3)X + d_1 = 0$$

and

$$p = \frac{y - (d_1 + d_2)}{y - x}.$$

– *If $\det(D) > 0$, $Z^D = p\epsilon_{(1-x)\otimes y} + (1-p)\epsilon_{(1-y)\otimes x}$ where x and y are the two solutions of the second degree polynomial equation*

$$X^2 - (2d_3 + d_4 + d_1)X + d_3 = 0$$

and

$$p = \frac{y - (d_3 + d_4)}{y - x}$$

The proof is quite involved and is provided in Sect. 4.1. Remark that each correlation system involves two points only in its support and that the parametrization of informationally optimal correlation systems involves three parameters, matching the dimension of X^{12}. Note that proposition 1 only proves the existence of optimal correlation systems with $\left|A^1\right| \cdot \left|A^2\right| + 1 = 5$ points in their support, thus described by 12 parameters.

3 Applications to repeated games with imperfect monitoring

A central problem in repeated games with imperfect monitoring is the generalization of the Folk theorem. This classical result asserts that if players perfectly observe the action profile and have high discount factors, then every feasible and individually rational payoff can be sustained by an equilibrium of the repeated game. An important issue is thus to find the individually rational level, i.e. the min max level of a player in a repeated game with imperfect monitoring. If all players but i want to punish player i, then they form a team of players who wish to correlate their actions in a way that is secret to player i. The connection to our concept is thus clear. The aim of this section is to show how to characterize the min max level through informationally optimal correlation and to use this characterization to solve examples.

3.1 The individually rational level in repeated games with imperfect monitoring

Let $N = \{1, \ldots, n\}$ be a team of players and $n + 1$ be another player. For each player $i \in N$, let A^i be player i's finite set of actions and let B be player $n + 1$'s finite set of actions. At each stage $t = 1, 2, \ldots$, each player chooses an action in his own set of actions and if $(a, b) = ((a^i)_{i\in N}, b) \in A \times B$ is the action profile played, the payoff for each team player $i \in N$ is $g(a, b)$ with $g\colon A \times B \to \mathbb{R}$ and the payoff for player $n + 1$ is $-g(a, b)$: for convenience we agree that team players are maximizing and player $n+1$ is minimizing. After each stage, if a is the action profile played by players $i \in N$, a signal s is drawn in a finite set S of signals with probability $q(s|a)$, where $q\colon A \to \Delta(S)$. Player $n + 1$ observes (s, b) and each player $i \in N$ observes (a, s, b): we consider games where all team members have the same information which contains the information of player $n + 1$.

A history of length t for the team is an element h_t of $H_t = (A\times B\times S)^t$, and a history of length t for player $n + 1$ is an element h_t^{n+1} of $H_t^{n+1} = (B \times S)^t$, by convention H_0 and H_0^{n+1} are singletons. A behavioral strategy σ^i for a team player i is a mapping $\sigma^i\colon \cup_{t\geq 0} H_t \to \Delta(A^i)$ and a behavioral strategy τ for player $n + 1$ is a mapping $\tau\colon \cup_{t\geq 0} H_t^{n+1} \to \Delta(B)$. A profile of behavioral strategies $(\sigma, \tau) = ((\sigma^i)_{i\in N}, \tau)$ induces a probability distribution $\mathbf{P}_{\sigma,\tau}$ on the set of plays $(A \times B \times S)^\infty$ endowed with the product σ-algebra.

Given a discount factor $0 < \lambda < 1$, the discounted payoff for the team induced by (σ, τ) is: $\gamma_\lambda(\sigma, \tau) = \mathbf{E}_{\sigma,\tau}[\sum_{t\geq 1}(1 - \lambda)\lambda^{t-1} g(\mathbf{a}_t, \mathbf{b}_t)]$ where $(\mathbf{a}_t, \mathbf{b}_t)$ denotes the random action profile at stage t. The λ-discounted max min payoff of the game denoted v_λ is:

$$v_\lambda = \max_\sigma \min_\tau \gamma_\lambda(\sigma, \tau)$$

The aim is to characterize and compute $\lim_{\lambda\to 1} v_\lambda$.

Fix a strategy of the team. At each stage t, player $n+1$, given his own history, holds a belief on the next action profile of the team, more precisely on the next profile of mixed strategy that the team will use. Therefore, player $n + 1$'s state of mind can be parameterized by a correlation system $Z = \sum_k p_k \epsilon_{\otimes x_k^i}$. Here k represents the whole past history h_t of the game up to stage t, and p_k the probability that player $n+1$ ascribes to it given his observations, i.e. $\mathbf{P}_{\sigma,\tau}(h_t|h_t^{n+1})$. How does the uncertainty of player $n+1$ evolve at the next stage? Before stage $t+1$, the uncertainty of player $n+1$ is measured by $H(\mathbf{k})$. Let $\mathbf{a}$ be the random action profile played by the team at stage $t+1$ and $\mathbf{s}$ be the random signal induced. Player $n+1$ observes neither $\mathbf{k}$ nor $\mathbf{a}$ but only $\mathbf{s}$. His new uncertainty is thus $H(\mathbf{k},\mathbf{a}|\mathbf{s})$. This leads to the following definition.

Definition 4 Let $Z = \sum_k p_k \epsilon_{\otimes_i x_k^i}$. Let $\mathbf{k}$ be a random variable with law (p_k), $\mathbf{a}$ be a random variable with values in A and with conditional distribution $\otimes_i x_k^i$ given $\{\mathbf{k} = k\}$ and let $\mathbf{s}$ be the induced random signal. The *entropy variation* associated to Z is

$$\Delta H(Z) = H(\mathbf{k},\mathbf{a}|\mathbf{s}) - H(\mathbf{k})$$

Now we relate $\Delta H(Z)$ with the expected entropy $J(Z)$. We recall the notion of mutual information: given $D \in \Delta(A)$, let $\mathbf{a}$ be a random action profile with distribution D and $\mathbf{s}$ be the induced random signal. The *mutual information* between $\mathbf{a}$ and $\mathbf{s}$ is

$$\begin{aligned} I_D(\mathbf{a},\mathbf{s}) &:= H(\mathbf{s}) - H(\mathbf{s}|\mathbf{a}) = H(\mathbf{a}) - H(\mathbf{a}|\mathbf{s}) \\ &= H\Big(\sum_a D(a) q(\cdot|a)\Big) - \sum_a D(a) H(q(\cdot|a)) \end{aligned}$$

It is a well defined and continuous function of the distribution D.

Lemma 2 *For each correlation system Z*

$$\Delta H(Z) = J(Z) - I_{D(Z)}(\mathbf{a},\mathbf{s})$$

Proof The chain rule for entropies gives

$$\begin{aligned} H(\mathbf{k},\mathbf{a},\mathbf{s}) &= H(\mathbf{s}) + H(\mathbf{k},\mathbf{a}|\mathbf{s}) \\ &= H(\mathbf{k}) + H(\mathbf{a},\mathbf{s}|\mathbf{k}) \\ &= H(\mathbf{k}) + H(\mathbf{a}|\mathbf{k}) + H(\mathbf{s}|\mathbf{a}) \end{aligned}$$

where the last equality holds since $\mathbf{s}$ is independent of $\mathbf{k}$ given $\mathbf{a}$. Therefore,

$$\begin{aligned} \Delta H(Z) &= H(\mathbf{a}|\mathbf{k}) + H(\mathbf{s}|\mathbf{a}) - H(\mathbf{s}) \\ &= J(Z) - I_{D(Z)}(\mathbf{a},\mathbf{s}) \end{aligned}$$

□

Gossner and Tomala [6] use these tools to characterize $\lim_\lambda v_\lambda$ as follows:

Theorem 2 (Gossner and Tomala [6]) *For $c \in \mathbb{R}$, let*

$$V(c) = \max_{Z:\Delta H(Z)\geq c} \min_b \mathbf{E}_{D(Z)}\, g(\mathbf{a}, b)$$

Then $\lim_\lambda v_\lambda$ *exists and,*

$$\lim_\lambda v_\lambda = \operatorname{cav} V(0)$$

with cav V *the smallest concave function pointwise (weakly) greater than* V.

We give an expression of $V(c)$ using informationally optimal correlation.

Proposition 3 *For* $c \in \mathbb{R}$, *let*

$$U(c) = \max_{D:\varphi(D) - I_D(\mathbf{a},\mathbf{s}) \geq c} \min_b \mathbf{E}_D\, g(\mathbf{a}, b)$$

Then, $V(c) = U(c)$.

Proof Since $\Delta H(Z) = J(Z) - I_{D(Z)}(\mathbf{a},\mathbf{s})$ and since Z is informationally optimal (io) if it maximizes $J(Z)$ under the constraints $D(Z) = D$,

$$U(c) = \max_{Z io:\Delta H(Z)) \geq c} \min_b \mathbf{E}_{D(Z)}\, g(\mathbf{a}, b)$$

thus $U(c) \leq V(c)$. Conversely, given any Z which is feasible for $V(c)$, one can replace Z by an informationally optimal system Z' such that $D(Z') = D(Z)$ without affecting $\min_b \mathbf{E}_{D(Z)}\, g(\mathbf{a}, b)$. □

3.2 A coordination game

We use Proposition 3 and Theorem 1 to give an explicit computation of the long run min max value for the following game. The team is $\{1, 2\}$ and plays against player 3. Players 1 and 2 both choose between spending the evening at the bar ‘Golden Gate’ (G) or at the bar ‘Happy Hours’ (H). Player 3 faces the same choice. The payoff for the team players is 1 if they meet at the same bar and 3 chooses the other bar, otherwise the payoff is 0. The payoff function is displayed below where 1 chooses the row, 2 the column and 3 the matrix.

$$\begin{array}{c} \\ G \\ H \\ \\ \end{array} \begin{array}{c} \quad G \;\; H \\ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\ G \end{array} \begin{array}{c} \quad G \;\; H \\ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ H \end{array}$$

The max min of the one-shot game in mixed strategies is $1/4$ and may be obtained in the repeated game by the team $\{1, 2\}$ by playing the same mixed action $(1/2, 1/2)$ at every stage. The max min in correlated strategies of the one-shot game is $1/2$. This may be obtained by players 1 and 2 in the repeated game if they can induce player 3 to believe, at almost every stage, that (G, G) and (H, H) will both be played with probability $1/2$ and if their play is independent on player 3’s behavior. For example, if player 3 has no information concerning the past moves of the opponents, then the team $\{1, 2\}$ may achieve its goal by randomizing evenly at the first stage, and coordinate all subsequent moves on the first action of player 1.

The case under study here is when player 3 observes the actions of player 2 but not of player 1, i.e. $S = A^2$ and $q(a'^2|a^1, a^2) = 1$ if $a'^2 = a^2$ and $q(a'^2|a^1, a^2) = 0$ otherwise. The study of this game with this signalling structure, which we denote Γ_0, was proposed by [14].

The following strategies for players 1 and 2 allow for partial correlation in the repeated game:

– At odd stages, play $\left(\frac{1}{2}, \frac{1}{2}\right) \otimes \left(\frac{1}{2}, \frac{1}{2}\right)$,
– at even stages, repeat the previous move of player 1. Player 3's belief is then that (G, G) is played with probability 1/2 and (H, H) with the same probability.

The limit time-average payoff yielded by this strategy is 3/8. Define two correlations systems as follows:

– $Z_{\frac{1}{2}} = \epsilon_{\frac{1}{2}\otimes\frac{1}{2}}$.
– $Z_1 = \frac{1}{2}\epsilon_{1\otimes 1} + \frac{1}{2}\epsilon_{0\otimes 0}$.

The distribution induced by Z_1 is $\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$. The distribution of signals under Z_1 puts weight 1/2 on both G and H thus $H(\mathbf{s}) = 1$. $H(\mathbf{s}|\mathbf{a}) = 0$ since the signal is a deterministic function of the action profile. For each k, $H(x_k) = H(y_k) = 0$, so $J(Z_1) = 0$. The entropy variation is $\Delta H(Z_1) = -1$. One has $J(Z_{\frac{1}{2}}) = 2$ and under Z_2, $H(\mathbf{s}) = 1$ and $H(\mathbf{s}|\mathbf{a}) = 0$, so $\Delta H(Z_{\frac{1}{2}}) = 1$.

The above strategy consists of playing $Z_{\frac{1}{2}}$ at odd stages and Z_1 at even stages, so that the team cyclically gains and loses 1 bit of entropy. If player 3 plays a best reply at each stage, the payoff obtained at odd stages is 1/4 and at even stages 1/2, thus in the long-run player 3 gets 3/8. How much correlation can be achieved by the team $\{1, 2\}$ in this game? Can the team improve on 3/8? Is it possible to achieve full correlation? We apply now our results to answer these questions.

Given $D = \begin{pmatrix} d_1 & d_2 \\ d_3 & d_4 \end{pmatrix}$, we let $\pi(D) = \min_b \mathbf{E}_D\, g(\mathbf{a}, b) = \min\{d_1, d_4\}$. We introduce a family of correlation systems of particular interest.

Notation 3 For $x \in [0, 1]$ let $Z(x) = \frac{1}{2}\epsilon_{x\otimes x} + \frac{1}{2}\epsilon_{(1-x)\otimes(1-x)}$.

It follows from Theorem 1 that each $Z(x)$ is informationally optimal. Actually, $(Z(x))_x$ is the family of informationally optimal correlation systems associated to probability measures that put equal weights on (G, G) and on (H, H), and equal weights on (G, H) and on (H, G). Against each $Z(x)$, player 3 is thus indifferent between his two actions and therefore,

$$\pi(D(Z(x))) = \frac{1}{2}(x^2 + (1-x)^2).$$

For each $k = 1, 2$, $H(x_k) = H(y_k) = h(x)$ and the law of signals under $Z(x)$ is $(\frac{1}{2}, \frac{1}{2})$ thus,

$$\Delta H(Z(x)) = 2h(x) - 1.$$

The following result, proved in Sect. 4.3, shows that the map U can be obtained from the family $(Z(x))_x$.

Proposition 4 *Consider the game Γ_0. For any $c \in [-1, 1]$,*

$$U(c) = \pi(D(Z(x_c))) = \frac{1}{2}(x_c^2 + (1 - x_c)^2)$$

with x_c the unique point in $\left[0, \frac{1}{2}\right]$ such that $2h(x_c) - 1 = c$. Moreover, U is concave.

It follows that the long-run max min for the game Γ_0 is $U(0)$.

Corollary 1 *For the game Γ_0, $\lim_\lambda v_\lambda$ is*

$$v = \frac{1}{2}(x_0^2 + (1 - x_0)^2)$$

where x_0 is the unique solution in $[0, \frac{1}{2}]$ of

$$-x \log(x) - (1 - x) \log(1 - x) = \frac{1}{2}$$

Numerically, $0.402 < v < 0.4021$.

Remark 1 In contrast with a finite zero-sum stochastic game, the max min here is transcendental. A similar property holds for the asymptotic value of a repeated game with incomplete information on both sides (see Mertens and Zamir [10]) and of a "Big Match" with incomplete information on one side (see Sorin [15]).

3.3 On the concavity/convexity of the map U

The function U is determined by the one-shot game and the signalling function. Since we deal with the computation of cav $U(0)$ two cases may arise: either cav $U(0) = U(0)$ (for example, if U is concave) or cav $U(0) > U(0)$ (if there exists two correlation systems Z_1, Z_2 and $0 < \lambda < 1$ s.t. $\lambda\pi(D(Z_1)) + (1 - \lambda)\pi(D(Z_2)) > U(0)$ and $\lambda \Delta H(Z_1) + (1 - \lambda)\Delta H(Z_2) \geq 0$).

In the previous section, we have shown that the map U corresponding to Γ_0 is concave. Goldberg [4] provides an example of the second case. Consider the game where payoffs for players 1 and 2 are given by the following matrices:

$$\begin{array}{c} \\ G \\ H \end{array} \begin{array}{c} \begin{array}{cc} G & H \end{array} \\ \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \\ G \end{array} \begin{array}{c} \begin{array}{cc} G & H \end{array} \\ \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \\ H \end{array}$$

The signals are deterministic and are given by the following matrix (they depends only on the moves of players 1 and 2):

$$\begin{array}{cc} & \begin{array}{cc} G & H \end{array} \\ \begin{array}{c} G \\ H \end{array} & \begin{pmatrix} s & s' \\ s'' & s \end{pmatrix} \end{array}$$

The max min in mixed strategies of the one-shot game is 5/4 and is obtained by the distribution $\frac{1}{2} \otimes \frac{1}{2}$. Allowing for correlation, the max min is 3/2 and is obtained by the distribution $\frac{1}{2} 0 \otimes 1 + \frac{1}{2} 1 \otimes 0$.

Relying on Theorem 1, Goldberg shows that U is convex so that its concavification is linear, thus $\operatorname{cav} U(0) = \frac{4}{3} = \frac{2}{3}\pi(D(Z')) + \frac{1}{3}\pi(D(Z''))$ where $Z' = \epsilon_{\frac{1}{2}\otimes\frac{1}{2}}$ and $Z'' = \frac{1}{2}\epsilon_{0\otimes 1} + \frac{1}{2}\epsilon 1 \otimes 0$.

4 Proofs of the main results

4.1 Proof of Theorem 1

For each integer m, let $C_m(D)$ be the set of set vectors $(p_k, x_k, y_k)_{k=1}^m$ where

$$\begin{cases} \forall k,\ p_k \geq 0, \quad \sum_{k=1}^m p_k = 1, x_k \in X^1, \quad y_k \in X^2 \\ \qquad\qquad\qquad \sum_{k=1}^m p_k x_k \otimes y_k = D \end{cases}$$

This set is clearly compact and the mapping

$$(p_k, x_k, y_k)_{k=1}^m \mapsto \sum_{k=1}^m p_k(H(x_k) + H(y_k))$$

is continuous on it. The problem (P_D) can thus be expressed as

$$\sup_m \max_{C_m(D)} \sum_{k=1}^m p_k(H(x_k) + H(y_k)) \quad (P_D)$$

Denote by $(P_{m,D})$, $m \geq 2$, the second maximization problem where m is fixed

$$\max_{C_m(D)} \sum_{k=1}^m p_k(h(x_k) + h(y_k)) \quad (P_{m,D})$$

4.1.1 Solving $(P_{2,D})$.

Given $D \in X^{12}$, a point in $C_2(D)$ is a vector $(p, (x_1, y_1), (x_2, y_2)) \in [0, 1]^5$ such that

$$D = p \begin{pmatrix} x_1 y_1 & x_1(1-y_1) \\ (1-x_1)y_1 & (1-x_1)(1-y_1) \end{pmatrix} + (1-p) \begin{pmatrix} x_2 y_2 & x_2(1-y_2) \\ (1-x_2)y_2 & (1-x_2)(1-y_2) \end{pmatrix}$$

The problem $(P_{2,D})$ is equivalent to

$$\max_{C_2(D)} p(h(x_1) + h(y_1)) + (1-p)(h(x_2) + h(y_2)) \quad (P_{2,D})$$

We are concerned with the computation of the set of solutions

$$\Lambda(D) := \operatorname{argmax}_{C_2(D)} p(h(x_1) + h(y_1)) + (1-p)(h(x_2) + h(y_2))$$

The problem $(P_{2,D})$ is the maximization of a continuous function on a compact set, thus $\Lambda(D) \neq \emptyset$ if $C_2(D) \neq \emptyset$. We will use the following parametrization: for $D = \begin{pmatrix} d_1 & d_2 \\ d_3 & d_4 \end{pmatrix}$, set $r = d_1 + d_2$, $s = d_1 + d_3$ and $t = d_1$. The vector $(p, (x_1, y_1), (x_2, y_2)) \in [0, 1]^5$ is in $C_2(D)$ if and only if:

$$\begin{cases} px_1 + (1-p)x_2 &= r \\ py_1 + (1-p)y_2 &= s \\ px_1y_1 + (1-p)x_2y_2 &= t \end{cases}$$

Note that $\det(D) := d_1d_4 - d_2d_3 = t - rs$.

The remainder of this section is devoted to the proof of the following characterization of $\Lambda(D)$:

Proposition 5 (A) *If* $\det(D) = 0$, *then*

$$\begin{aligned} \Lambda(D) = &\{(p, (r, s), (r, s)) : p \in [0, 1]\} \\ &\cup \left\{(1, (r, s), (y_1, y_2)) : (y_1, y_2) \in [0, 1]^2\right\} \\ &\cup \left\{(0, (x_1, x_2), (r, s)) : (x_1, x_2) \in [0, 1]^2\right\} \end{aligned}$$

(B) *If* $\det(D) < 0$,

$$\Lambda(D) = \left\{\left(\frac{\beta - r}{\beta - \alpha}, (\alpha, \beta), (\beta, \alpha)\right); \left(\frac{\alpha - r}{\alpha - \beta}, (\beta, \alpha), (\alpha, \beta)\right)\right\}$$

where α *and* β *are the two solutions of:*

$$X^2 - (2d_1 + d_2 + d_3)X + d_1 = 0.$$

(C) *If* $\det(D) > 0$,

$$\begin{aligned} \Lambda(D) = &\left\{\left(\frac{\beta - (1-r)}{\beta - \alpha}, (1-\alpha, \beta), (1-\beta, \alpha)\right);\right. \\ &\left.\left(\frac{\alpha - (1-r)}{\alpha - \beta}, (1-\beta, \alpha), (1-\alpha, \beta)\right)\right\} \end{aligned}$$

where α and β are the two solutions of

$$X^2 - (2d_3 + d_4 + d_1)X + d_3 = 0.$$

Remark that in each case all solutions correspond to the same correlation system. Solutions of $(P_{2,D})$ thus always lead to a unique correlation system.

Point (A). The formula given in proposition 5 for $\Lambda(D)$ clearly defines a subset of $C_2(D)$. Note that $\det(D) = 0$ if and only if $D = r \otimes s$. (A) follows then directly from point (3) of lemma 2.

Points (B) and (C). First we show that these cases are deduced from one another by symmetry. Take a distribution $D = \begin{pmatrix} d_1 & d_2 \\ d_3 & d_4 \end{pmatrix}$ and a point $(p, (x_1, y_1), (x_2, y_2))$ in $\Lambda(D)$. Let then $D' = \begin{pmatrix} d_3 & d_4 \\ d_1 & d_2 \end{pmatrix}$ and remark that

- $\det(D') = -\det(D)$
- $(p, (1 - x_1, y_1), (1 - x_2, y_2)) \in \Lambda(D')$.

Remark also that the two solutions given in Proposition 5 for case (C) are deduced from the solutions for case (B) by symmetry. We thus need to prove (B) only.

Since α and β are solutions of:

$$X^2 - (2d_1 + d_2 + d_3)X + d_1 = 0.$$

we have $\alpha + \beta = r + s$ and $\alpha\beta = t$. Thus α, β, $\frac{\beta - r}{\beta - \alpha}$ and $\frac{\alpha - r}{\alpha - \beta}$ are in [0, 1]. One then easily verifies that:

$$\begin{cases} \frac{\beta-r}{\beta-\alpha}\alpha + \frac{\alpha-r}{\alpha-\beta}\beta & = r \\ \frac{\beta-r}{\beta-\alpha}\beta + \frac{\alpha-r}{\alpha-\beta}\alpha & = s \\ \frac{\beta-r}{\beta-\alpha}\alpha\beta + \frac{\alpha-r}{\alpha-\beta}\beta\alpha & = r \end{cases}$$

The solutions given in proposition 5 for case (B) are thus in $C_2(D)$ which is therefore non empty. In particular, any 2×2 joint distribution can be decomposed as a convex combination of two independent distributions.

We solve now the case where D is in the boundary of X^{12}.

Case 1 D is in the boundary

Assuming $\det(D) < 0$, we get either

$$D = D_1 = \begin{pmatrix} 0 & r \\ s & 1 - r - s \end{pmatrix}$$

or

$$D = D_2 = \begin{pmatrix} 1-r-s & s \\ & \\ r & 0 \end{pmatrix}$$

with $rs > 0$. We solve for D_1, the other case being similar. The vector $(p, (x_1, y_1), (x_2, y_2))$ is in $\Lambda(D_1)$ if and only if

$$\begin{aligned} px_1 + (1-p)x_2 &= r \\ py_1 + (1-p)y_2 &= s \\ px_1y_1 + (1-p)x_2y_2 &= 0 \end{aligned}$$

Since D is not the product of its marginals, necessarily $p \in (0, 1)$, and $x_1y_1 = x_2y_2 = 0$. We assume wlog. $x_1 = 0$. We get then $x_2 = \frac{r}{1-p} \neq 0$, $y_2 = 0$, and $y_1 = \frac{s}{p}$. The problem (P_{2,D_1}) is then reduced to maximizing the expression over $p \in (0, 1)$

$$ph\left(\frac{s}{p}\right) + (1-p)h\left(\frac{r}{1-p}\right)$$

A solution in (0, 1) exists, from the non emptiness of $\Lambda(D_1)$. The first order condition writes

$$h\left(\frac{s}{p}\right) - \frac{s}{p}h'\left(\frac{s}{p}\right) = h\left(\frac{r}{1-p}\right) - \frac{r}{1-p}h'\left(\frac{r}{1-p}\right)$$

The map $f\colon (0, 1) \to \mathbb{R}$ given by $f(x) = h(x) - xh'(x)$ has derivative $f'(x) = -xh''(x) > 0$, hence is strictly increasing. Thus, the first order condition is equivalent to $\frac{r}{1-p} = \frac{s}{p}$, or $p = \frac{s}{r+s}$. We have thus shown

$$\Lambda(D_1) = \left\{\left(\frac{s}{r+s}, 0, r+s, r+s, 0\right); \left(\frac{r}{r+s}, r+s, 0, 0, r+s\right)\right\}$$

Case 2 D is interior

We assume now that $\min_{i\in\{1,\dots,4\}}(d_i) > 0$. The proof is organized in a series of lemmata. Lemma 3 proves that all solutions are interior. Therefore they must verify a first order condition. First order equations are established in lemma 4. Lemma 5 studies the solutions of the first order equations and lemma 6 shows uniqueness of those solutions. We conclude the proof with lemma 7.

We prove now that any solution of $(P_{2,D})$ is interior. This is due to the fact that the entropy function has infinite derivative at the boundary.

Lemma 3 *If* $\min_{i\in\{1,\dots,4\}}(d_i) > 0$ *and* $\det(D) \neq 0$ *then* $\Lambda(D) \subset (0, 1)^5$.

Proof We prove that elements of $\Lambda(D)$ are interior. Take a point $Z = (p, (x_1, y_1), (x_2, y_2))$ in $C_2(D)$. Since $\det(D) \neq 0$, $0 < p < 1$. We show that if $x_1 = 0$, Z is not optimal for $(P_{2,D})$. The proof is completed by symmetry. We assume thus $x_1 = 0$ and construct a correlation system $Z^\varepsilon = (p^\varepsilon, (x_1^\varepsilon, y_1^\varepsilon), (x_2^\varepsilon, y_2^\varepsilon))$ in $C_2(D)$ as follows. Since $Z \in C_2(D)$

$$\begin{cases} (1-p)x_2 & = r \\ py_1 + (1-p)y_2 & = s \\ (1-p)x_2 y_2 & = t \end{cases}$$

Take $\varepsilon > 0$ and let

$$\begin{cases} p^\varepsilon = & p + \varepsilon \\ x_1^\varepsilon = & \left(1 - \frac{p}{p^\varepsilon}\right) x_2 \\ x_2^\varepsilon = & x_2 \\ y_1^\varepsilon = & y_1 \\ y_2^\varepsilon = & \frac{1-p}{1-p^\varepsilon} y_2 - \frac{p^\varepsilon - p}{1-p^\varepsilon} y_1 \end{cases}$$

Since $t = (1-p)x_2 y_2 \neq 0$, there exists $\varepsilon_0 > 0$ such that $Z^\varepsilon \in [0,1]^5$ for $0 < \varepsilon \le \varepsilon_0$. A simple computation shows that Z^ε is in $C_2(D)$. We now compare the objective function of $(P_{2,D})$ at Z^ε and at Z.

$$\begin{aligned}
&\left(p^\varepsilon \left(h(x_1^\varepsilon) + h(y_1^\varepsilon)\right) + (1-p^\varepsilon)\left(h(x_2^\varepsilon) + h(y_2^\varepsilon)\right)\right) \\
&\quad - (p\,[h(x_1) + h(y_1)] + (1-p)\,[h(x_2) + h(y_2)]) \\
&= qh(x_1^\varepsilon) + (1-p^\varepsilon)h(y_2^\varepsilon) - (1-p)h(y_2) \\
&= (p+\varepsilon)h\left(\left(1 - \frac{p}{p+\varepsilon}\right) x_2\right) \\
&\quad + (1-p-\varepsilon)h\left(\frac{1-p}{1-p-\varepsilon} y_2 - \frac{\varepsilon}{1-p-\varepsilon} y_1\right) - (1-p)h(y_2) \\
&= ph\,(\varepsilon x_2) + (1-p)h\left(y_2 - \frac{\varepsilon}{1-p} y_1\right) - (1-p)h(y_2) + o(\varepsilon) \\
&= ph(\varepsilon x_2) - \varepsilon y_1 h'(y_2) + o(\varepsilon) \\
&= p\,[-\varepsilon x_2 \ln(\varepsilon x_2) - (1-\varepsilon x_2)\ln(1-\varepsilon x_2)] - \varepsilon y_1 h'(y_2) + o(\varepsilon) \\
&= \varepsilon\left[-p x_2 \ln(\varepsilon x_2) - y_1 h'(y_2) + x_2 + o(1)\right] \\
&> 0
\end{aligned}$$

for ε small enough. □

Solutions of $(P_{2,D})$ being interior, they must satisfy the first order conditions. Given x and y in $(0,1)$, recall that the Kullback distance $d_K(x \,\|\, y)$ of x with respect to y is defined by

$$d_K(x \,\|\, y) = x \log \frac{x}{y} + (1-x) \log \frac{1-x}{1-y}$$

A direct computation shows

$$d_K(x \,\|\, y) = h(y) - h(x) - h'(y)(y - x),$$

where h' denotes the derivative of h.

Lemma 4 *Suppose that* $\min_i(d_i) > 0$ *and* $\det(D) \neq 0$. *If* $(p, x_1, y_1, x_2, y_2) \in \Lambda(D)$ *then:*

$$\begin{cases} d_K(x_2 \,\|\, x_1) = d_K(y_1 \,\|\, y_2) \\ d_K(x_1 \,\|\, x_2) = d_K(y_2 \,\|\, y_1) \end{cases} \quad (E)$$

Proof The Lagrangian of $(P_{2,D})$ writes

$$\begin{aligned} \mathcal{L}(p, x_1, y_1, x_2, y_2, \alpha, \beta, \gamma) = \; & p(h(x_1) + h(y_1)) + (1 - p)(h(x_2) + h(y_2)) \\ & + \alpha(px_1 + (1 - p)x_2 - r) + \beta(py_1 + (1-p)y_2 - s) \\ & + \gamma(px_1y_1 + (1 - p)x_2y_2 - t) \end{aligned}$$

The partial derivatives are

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial p} = (h(x_1) + h(y_1)) - (h(x_2) + h(y_2)) + \alpha(x_1 - x_2) + \beta(y_1 - y_2) \\ \qquad + \gamma(x_1y_1 - x_2y_2) \\ \frac{\partial \mathcal{L}}{\partial x_1} = p\left(h'(x_1) + \alpha + \gamma y_1\right) \\ \frac{\partial \mathcal{L}}{\partial x_2} = (1 - p)\left(h'(x_2) + \alpha + \gamma y_2\right) \\ \frac{\partial \mathcal{L}}{\partial y_1} = p\left(h'(y_1) + \beta + \gamma x_1\right) \\ \frac{\partial \mathcal{L}}{\partial y_2} = (1 - p)\left(h'(y_2) + \beta + \gamma x_2\right) \end{cases}$$

If $(p, x_1, y_1, x_2, y_2) \in \Lambda(D)$, there exists (α, β, γ) such that

$$\begin{cases} (h(x_1) + h(y_1)) - (h(x_2) + h(y_2)) + \alpha(x_1 - x_2) & \\ \quad +\beta(y_1 - y_2) + \gamma(x_1y_1 - x_2y_2) = 0 & (E1) \\ h'(x_1) + \alpha + \gamma y_1 = 0 & (E2) \\ h'(x_2) + \alpha + \gamma y_2 = 0 & (E3) \\ h'(y_1) + \beta + \gamma x_1 = 0 & (E4) \\ h'(y_2) + \beta + \gamma x_2 = 0 & (E5) \end{cases}$$

The combination of equations $(E1) - x_1 \times (E2) + x_2 \times (E3)$ gives

$$(h(x_1) + h(y_1)) - (h(x_2) + h(y_2)) = x_1h'(x_1) - x_2h'(x_2) - \beta(y_1 - y_2) \quad (1)$$

The combination $y_1((E4) - (E5)) + (x_1 - x_2)(E_2)$ writes

$$y_1\left(h'(y_1) - h'(y_2)\right) = h'(x_1)(x_1 - x_2) + \alpha(x_1 - x_2) \quad (2)$$

Equations (1) and (2) give

$$h(x_1) - h(x_2) - h'(x_1)(x_1 - x_2) = h(y_2) - h(y_1) - h'(y_2)(y_2 - y_1)$$

which rewrites

$$d_K(x_2 \| x_1) = d_K(y_1 \| y_2)$$

Similarly we obtain

$$d_K(x_1 \| x_2) = d_K(y_2 \| y_1)$$

□

We give now the solutions of the equations (E).

Lemma 5 *Assume $d_K(x \| a) = d_K(b \| y)$ and $d_K(a \| x) = d_K(y \| b)$. Then one of the following holds:*

(F1) $x = b,\ y = a;$
(F2) $x = 1 - b,\ y = 1 - a;$
(F3) $x = a,\ y = b.$

Proof Fix a and b in $(0, 1)$. We need to solve the system

$$\begin{cases} d_K(x \| a) - d_K(b \| y) = 0 \\ d_K(a \| x) - d_K(y \| b) = 0 \end{cases} \quad (S)$$

It is immediate to check that $(F1)$, $(F2)$, and $(F3)$ are solutions of (S). Letting $S(x, y) = (d_K(x \| a) - d_K(b \| y), d_K(a \| x) - d_K(y \| b))$, the Jacobian $J(x, y)$ of S writes:

$$\begin{aligned} J(x, y) &= \det \begin{pmatrix} \ln(\frac{x}{1-x}) - \ln(\frac{a}{1-a}) & \frac{1-a}{1-x} - \frac{a}{x} \\ \frac{1-b}{1-y} - \frac{b}{y} & \ln(\frac{y}{1-y}) - \ln\left(\frac{b}{1-b}\right) \end{pmatrix} \\ &= \ln \frac{x(1-a)}{a(1-x)} \times \ln \frac{y(1-b)}{b(1-y)} - \frac{(x-a) \times (y-b)}{x(1-x)y(1-y)} \end{aligned}$$

since for all $z > 1$, $0 < \ln(z) < z - 1$, if $x > a$ and $y > b$ then

$$0 < \ln \frac{x(1-a)}{a(1-x)} < \frac{x(1-a)}{a(1-x)} - 1 = \frac{x-a}{1-x} < \frac{x-a}{x(1-x)}$$

and

$$0 < \ln \frac{y(1-b)}{b(1-y)} < \frac{y(1-b)}{b(1-y)} - 1 = \frac{y-b}{1-y} < \frac{y-b}{y(1-y)}$$

Hence, on the domain $\{x > a, y > b\}$ one has

$$\ln \frac{x(1-a)}{a(1-x)} \times \ln \frac{b(1-y)}{y(1-b)} < \frac{x-a}{1-x} \times \frac{y-b}{1-y} < \frac{(x-a) \times (b-y)}{x(1-x)y(1-y)}.$$

Thus $J(x, y) < 0$ on the domain $\{x > a, y > b\}$. The mappings $x \mapsto d_K(x \| a) := f_a(x)$ and $y \mapsto d_K(b \| y) := g_b(y)$ are differentiable and strictly increasing on the intervals $(a, 1)$ and $(b, 1)$, respectively, and setting $F(x) := g_b^{-1} \circ f_a(x) - f_b^{-1} \circ g_a(x)$, $S(x, y) = 0$ if and only if $F(x) = 0$ and $y = g_b^{-1} \circ f_a(x)$. Then if $x_0 \in (a, 1)$ is such that $F(x_0) = 0$, we let $y_0 := g_b^{-1} \circ f_a(x_0) = f_b^{-1} \circ f_a(x_0) \in (b, 1)$ and $F'(x_0) = \frac{J(x_0, y_0)}{f_b'(y_0) \times g_b'(y_0)} < 0$, *i.e.* at a zero of F, $F'(x_0) < 0$. F admits thus at most one zero.
If $a + b < 1$, $(1 - b, 1 - a)$ is indeed a solution of (S) and we deduce

D_1. *If $a + b < 1$, then $(1 - b, 1 - a)$ is the unique solution of (S) on $\{x > a, y > b\}$.*
Using $z - 1 < \ln(z) < 0$ for all $z < 1$, we deduce that $J(x, y) < 0$ on the domain $\{x < a,\ y < b\}$. We then obtain
D_2. *If $a + b > 1$, then $(1 - b, 1 - a)$ is the unique solution of (S) on $\{x < a,\ y < b\}$.*
Similar arguments show that
D_3. *If $a < b$, then (b, a) is the unique solution to (S) on $\{x > a,\ y < b\}$.*
D_4. *If $a > b$, then (b, a) is the unique solution to (S) on $\{x < a,\ y > b\}$.*

We are now in position to complete the proof of the lemma. First, if $(x - a)(y - b) = 0$ then (S) implies $x = a$ and $y = b$.
If $(x - a)(y - b) > 0$, we obtain $(x, y) = (1 - b, 1 - a)$ as follows:

- If $a + b \le 1$:
  - If $x < a$ and $y < b$ then $x + y < a + b \le 1$. Apply D_1 reversing the roles of (x, y) and (a, b).
  - If $x > a$, $y > b$ and $a + b \neq 1$. Apply D_1.
  - If $x > a$, $y > b$ and $a + b = 1$ then $x + y > 1$. Apply D_2, reversing the roles.
- If $a + b > 1$:
  - If $x > a$ and $y > b$, then $x + y > a + b > 1$. Apply D_2, reversing the roles.
  - If $x < a$ and $y < b$, apply D_2.

If $(x - a)(y - b) < 0$ we obtain $(x, y) = (b, a)$ as follows:

- If $a \le b$:
  - If $x < a$ and $y > b$ then $x < y$. Reverse the roles and apply D_3.
  - If $x > a$, $y < b$ and $a < b$, apply D_3.
  - If $x > a$, $y < b$ and $a = b$ then $x > y$. Reverse the roles and apply D_4.
- If $a > b$:
  - If $x > a$ and $y < b$ then $x > y$. Reverse the roles and apply D_4.
  - If $x < a$ and $y > b$, apply D_4. □

Lemma 6 1. *If* $\det(D) < 0$, *solutions of* $(P_{2,D})$ *are of type* $(F1)$.
2. *If* $\det(D) > 0$, *solutions of* $(P_{2,D})$ *are of type* $(F2)$.
3. *If* $\det(D) = 0$, *solutions of* $(P_{2,D})$ *are of type* $(F3)$.

Proof Let $(p, a, b) \in [0, 1]^3$, it is straightforward to check that

1. $\det[p(a \otimes b) + (1-p)(b \otimes a)] \leq 0$
2. $\det[p(a \otimes b) + (1-p)[1-b] \otimes [1-a]] \geq 0$

The result follows then directly from lemma 5. □

We now conclude the proof of proposition 5

Lemma 7 *Let D such that* $\det(D) < 0$. *Then*

$$\Lambda(D) = \left\{ \left(\frac{\beta - r}{\beta - \alpha}, \alpha, \beta, \beta, \alpha \right); \left(\frac{r - \alpha}{\beta - \alpha}, \beta, \alpha, \alpha, \beta \right) \right\}$$

where α *and* β *are the two solutions of the equation* $X^2 - (r+s)X + t = 0$.

Proof Assuming $\det(D) < 0$, it follows from lemma 6 that any element of $\Lambda(D)$ is a tuple $(p, (x, y), (y, x))$, with

$$\begin{cases} px + (1-p)y &= r \\ py + (1-p)x &= s \\ pxy + (1-p)yx &= t \end{cases}$$

We deduce then

$$\begin{cases} x + y = r + s \\ xy \ = \ t \end{cases}$$

so that x and y must be solutions of the equation: $X^2 - (r+s)X + t = 0$ and p is given by $p = \frac{y-r}{y-x}$. Note that

$$\Delta = (r+s)^2 - 4t \geq 4(rs - t) = -4\det(D) > 0$$

Hence, this equation admits two distinct solutions α and β. □

The proof of proposition 5 is thus complete.

4.2 Solving $(P_{m,D})$

To conclude the proof of Theorem 1, we prove that for every $D \in X^{12}$, the value of $P_{m,D}$, $m > 2$ and of $P_{2,D}$ are the same. Recall from Lemma 1 that if $(p_k, x_k, y_k)_{k \in K}$ is optimal for $P_{m,D}$, then for any pair (k_1, k_2) s.t. $p_{k_1} + p_{k_2} > 0$, the correlation system $\left(\left(\frac{p_{k_1}}{p_{k_1} + p_{k_2}}, x_{k_1}, y_{k_1} \right); \left(\frac{p_{k_2}}{p_{k_1} + p_{k_2}}, x_{k_2}, y_{k_2} \right) \right)$ is optimal for the distribution it induces. We deduce the solutions of $(P_{m,D})$ and of (P_D) from the form of solutions of $(P_{2,D})$

Lemma 8 *Let* $(p_k, x_k, y_k)_{k=1}^m \in C_m(D)$ *such that for all* k, $p_k > 0$.

If $(p_k, x_k, y_k)_{k=1}^m$ *is optimal for* (P_D) *then one of the following holds*

- $\forall k$, *if* $(x_k, y_k) \neq (x_1, y_1)$ *then* $(x_k, y_k) = (y_1, x_1)$
- $\forall k$, *if* $(x_k, y_k) \neq (x_1, y_1)$ *then* $(x_k, y_k) = (1 - y_1, 1 - x_1)$

Proof Suppose that $(x_2, y_2) \neq (x_1, y_1)$. Since $(p_k, x_k, y_k)_{k=1,\ldots,m}$ is optimal for (P_D), $\left(\left(\frac{p_1}{p_1+p_2}, x_1, y_1\right), \left(\frac{p_2}{p_1+p_2}, x_2, y_2\right)\right)$ is an optimal correlation system. Then one has either $(x_2, y_2) = (y_1, x_1)$ or $(x_2, y_2) = (1 - y_1, 1 - x_1)$. Suppose wlog. that $(x_2, y_2) = (y_1, x_1)$. Let us prove that if $(x_k, y_k) \neq (x_1, y_1)$ then we have also $(x_k, y_k) = (y_1, x_1)$. If it is not the case, we must have $(x_k, y_k) = (1 - y_1, 1 - x_1)$. Thus we deduce that $(x_k, y_k) = (1 - x_2, 1 - y_2)$. This is compatible with the form of optimal correlation system (with $m = 2$), only if we have either $(1 - x_2, 1 - y_2) = (1 - y_2, 1 - x_2)$ or $(1 - x_2, 1 - y_2) = (y_2, x_2)$. This means that we must assume either $x_2 = y_2$ or $x_2 = 1 - y_2$. If $x_2 = y_2$ then, since $(x_2, y_2) = (y_1, x_1)$, we should have $x_1 = y_1$. This implies that $(x_2, y_2) = (x_1, y_1)$, a contradiction with our assumption that $(x_2, y_2) \neq (x_1, y_1)$. Now, if $x_2 = 1 - y_2$ we deduce that $(x_k, y_k) = (y_2, x_2)$ from which we get $(x_k, y_k) = (x_1, y_1)$, also in contradiction with our assumption. Hence, if $(x_2, y_2) = (y_1, x_1)$ then $\forall k$, if $(x_k, y_k) \neq (x_1, y_1)$ one has $(x_k, y_k) = (y_1, x_1)$. □

This ends the proof of Theorem 1.

4.3 Proof of Proposition 4

We use Theorem 1 to solve the problem

$$U(c) = \max_{D:\varphi(D) - I_D(\mathbf{a},\mathbf{s}) \geq c} \pi(D)$$

for the game Γ_0.

Definition 5 A correlation system Z is dominated for Γ_0 if there exists Z' such that $\pi(D(Z')) \geq \pi(D(Z))$ and $\Delta H(Z') \geq \Delta H(Z)$ with at least one strict inequality. Z is undominated otherwise.

From Theorem 1, undominated correlation systems must be of the form $p\epsilon_{x\otimes y} + (1-p)\epsilon_{y\otimes x}$ or $p\epsilon_{x\otimes y} + (1-p)\epsilon_{1-y\otimes 1-x}$. The next lemma shows that the first family of solutions is dominated.

Lemma 9 *Given* $Z = p\epsilon_{x\otimes y} + (1-p)\epsilon_{y\otimes x}$, *let* $Z' = \epsilon_{x\otimes y}$ *and* $Z'' = \epsilon_{y\otimes x}$. *Then:*

1. $\pi(D(Z)) = \pi(D(Z')) = \pi(D(Z''))$
2. $\Delta H(Z) \leq \max(\Delta H(Z'), \Delta H(Z''))$ *with strict inequality if* $x \neq y$ *and* $0 < p < 1$.

Proof For point (1), the common value is $\min(xy, (1-x)(1-y))$. Point (2) follows from the formulas $\Delta H(Z) = h(x) + h(y) - h(px + (1-p)y)$, $\Delta H(Z') = h(x) + h(y) - h(x)$, $\Delta H(Z'') = h(y) + h(x) - h(y)$ and the strict concavity of h. □

We search now solutions among the family of optimal correlation systems $p\epsilon_{x\otimes y} + (1-p)\epsilon_{1-y\otimes 1-x}$.

Lemma 10 *Let* $Z = p\epsilon_{x\otimes y} + (1-p)\epsilon_{1-y\otimes 1-x}$, $0 < p < 1$ *and* $x \neq 1 - y$. *If* Z *is undominated for* Γ_0, *then* $p = \frac{1}{2}$.

Proof Denote the distribution induced by Z, $D(Z) = \begin{pmatrix} d_1(Z) & d_2(Z) \\ d_3(Z) & d_4(Z) \end{pmatrix}$

Assuming $x \neq 1 - y$, $p = \frac{1}{2}$ is equivalent to $d_1(Z) = d_4(Z)$. Assume by contradiction that $d_1(Z) \neq d_4(Z)$ and by symmetry $d_1(Z) < d_4(Z)$. The Lagrangian of the maximization problem,

$$\max_{\begin{cases} Z = ((p, x, y); (1-p, 1-y, 1-x)) \\ \Delta H(Z) \geq c \end{cases}} \pi(D(Z))$$

writes

$$\mathcal{L} = pxy + (1-p)(1-x)(1-y) - \alpha(h(x) + h(y) - h(px + (1-p)(1-y)))$$

Let $\tilde{y} = 1 - y$ and $z = px + (1-p)\tilde{y}$:

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial p} = & (x - \tilde{y})(1 - \alpha h'(z)) \\ \frac{\partial \mathcal{L}}{\partial x} = & -\tilde{y} + p + \alpha(h'(x) - ph'(z)) \\ \frac{\partial \mathcal{L}}{\partial y} = & x - 1 + p + \alpha(-h'(\tilde{y}) + (1-p)ph'(z)) \end{cases}$$

so that optimality of Z implies:

$$\begin{cases} h'(z) = \frac{1}{\alpha} \\ \tilde{y} = \frac{h'(x)}{h'(z)} \\ x = \frac{h'(\tilde{y})}{h'(z)} \end{cases}$$

From the first two conditions we deduce that $h'(x)h'(\tilde{y}) \geq 0$, hence x and $\tilde{y}$ lie on the same side of $1/2$. But then $|h'(z)| \geq |h'(x)|$ and $|h'(z)| \geq |h'(\tilde{y})|$ is inconsistent with z lying in the strict interval of extremities x and $\tilde{y}$: $0 < p < 1$, $x \neq \tilde{y}$. □

Lemma 11 *Let $Z = \frac{1}{2}\epsilon_{x \otimes y} + \frac{1}{2}\epsilon_{1-y \otimes 1-x}$, with $x \neq 1 - y$. If Z is not dominated for Γ_0, then $x = y$.*

Proof Let $z = \frac{x+y}{2}$, and $Z' = \left(\left(\frac{1}{2}, z, z\right), \left(\frac{1}{2}, 1-z, 1-z\right)\right)$. We prove that Z' dominates Z in G if $x \neq y$. For payoffs, direct computation leads $\pi(D(Z')) - \pi(D(Z)) = (\frac{x+y}{2})^2$. For entropy variations, let ψ be defined by $\psi(x, y) = h(x) + h(y) - h(\frac{x+1-y}{2})$. Then $\Delta H(Z) = \psi(x, y) = \psi(y, x)$ and $\Delta H(Z') = \psi\left(\frac{x+y}{2}, \frac{x+y}{2}\right)$. Inequality $\psi\left(\frac{x+y}{2}, \frac{x+y}{2}\right) > \frac{\psi(x,y)+\psi(y,x)}{2}$ will follow from the strict concavity of ψ. The Jacobian matrix of ψ is

$$J = \begin{pmatrix} h''(x) - \frac{1}{4}h''(\frac{x+1-y}{2}) & -\frac{1}{4}h''(\frac{x+1-y}{2}) \\ -\frac{1}{4}h''(\frac{x+1-y}{2}) & h''(y) - \frac{1}{4}h''(\frac{x+1-y}{2}) \end{pmatrix}$$

Then, trace$J = h''(x) + h''(y) - \frac{1}{2}h''\left(\frac{x+1-y}{2}\right) = h''(x) + h''(1-y) - \frac{1}{2}h''\left(\frac{x+1-y}{2}\right)$ is negative since $h'' : t \mapsto -\frac{1}{\ln 2}\left(\frac{1}{t} + \frac{1}{1-t}\right)$ is both concave and negative on (0, 1). Computation of det J shows:

$$\det J = \frac{1}{(\ln 2)^2} \frac{(1-x)(1-y) + xy}{xy(1-x)(1-y)(1-x+y)(1-y+x)} > 0$$

Hence the strict concavity of ψ, and the claim follows. □

We prove now Proposition 4. From the two previous lemmas, it follows that an undominated correlation system is of the form $Z(x) = \frac{1}{2}\epsilon_{x\otimes x} + \frac{1}{2}\epsilon_{1-x\otimes 1-x}$ with $x \in [0, 1]$. The graph of Fig. 1 $c \mapsto U(c)$ is thus the set

$$C = \left\{(\Delta H(Z), \pi(D(Z))), Z = \frac{1}{2}\epsilon_{x\otimes x} + \frac{1}{2}\epsilon_{1-x\otimes 1-x} \text{ and } x \in [0, 1]\right\}$$

By symmetry one needs only to consider to $x \in [0, \frac{1}{2}]$, and letting $(s(x), t(x)) = (2h(x) - 1, \frac{1}{2}x^2 + \frac{1}{2}(1-x)^2)$, C is the parametric curve $\{(s(x), t(x)), x \in [0, \frac{1}{2}]\}$. Since the slope $\alpha(x)$ of C at $(s(x), t(x))$ is

$$\alpha(x) = \frac{\mathrm{d}t(x)/\mathrm{d}x}{\mathrm{d}s(x)/\mathrm{d}x} = \frac{1-2x}{\log(1-x) - \log(x)}$$

and

$$\alpha'(x) = \frac{2x - 1 + 2x(1-x)\ln(1-\frac{1}{x})}{\ln(2)x(1-x)(\log(1-x) - \log(x))^2}$$

The numerator of this expression has derivative $(1-2x)\ln\left(\frac{1}{x} - 1\right) > 0$, and takes the value 0 at $x = \frac{1}{2}$, hence it is nonnegative and so is $\alpha'(x)$. We conclude that C is concave

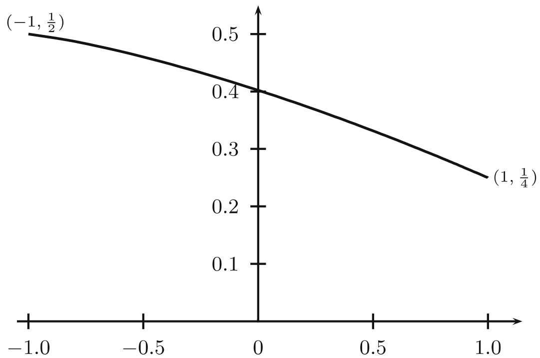

Fig. 1 The graph of U

and that $U(c) = \pi(D(Z(x_c)))$ with $\Delta H(Z(x_c)) = 2h(x_c) - 1 = c$ and $\operatorname{cav} U(0) = U(0)$. This value is $\frac{1}{2}x^2 + \frac{1}{2}(1-x)^2$, where $0 < x < 1$ solves $h(x) = \frac{1}{2}$. Numerical resolution yields $0.1100 < x < 0.1101$ and $0.4020 < \frac{1}{2}x^2 + \frac{1}{2}(1-x)^2 < 0.4021$.

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