The Value of Information in Zero-Sum Games

Olivier Gossner
THEMA, Université Paris Nanterre
and CORE, Université Catholique de Louvain

joint work with Jean-François Mertens

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Abstract:

It is not true that the value of information is positive in economic situations, as shown for instance by the Hirshleifer (1971) paradoxes. Recall the following example of information rejection, which is a variant of the example provided by Kamien, Tauman and Zamir (1986). A card in a deck can be white or black, with equal probabilities. Two players, player 1 and 2 are sequentially asked to guess the color of the card. A player gets a payoff of 6 if he/she is the only one to guess right, both players get a value of 2 if both guess right, and a player guessing wrong gets 0. Assume that no player knows the color of the card, then 1 announces a color, and 2's best response is to announce the other color. Both players get an expected payoff of 3. Now, assume that player 1 gets to see the card before making his/her announcement (and player 2 sees that player 1 sees it). Then, player 1's dominant strategy is to announce the true color, and player 2 is better off making the same announcement as 1: both players get a payoff of 2.

For one agent, the value of information is known to be positive: more information is always better. Actually, Blackwell's theorem (1951, 1953) says more than this. Blackwell defines two notions of comparison of statistical experiments. Payoff-wise, a statistical experiment is better than another one if and only if it yields a better payoff in every decision problem. Information-wise, a statistical experiment is more informative than another one if and only if the information of the latter can be obtained by garbling the information of the former. Blackwell's theorem establishes the equivalence between the two notions.

Zero-sum games represent perfectly antagonistic situations between two players, 1 and 2. The payoff received by player 1 is paid to him/her by player 2. This class of games are important for a number of reasons. They were the first class of games studied by von-Neumann and Morgenstern (1944). They used zero-sum games as a basis in their study of general non-zero-sum games. In fact, many results in the zero-sum theory are useful for the non-zero-sum theory: think of the characterization of punishment levels in repeated games, or the existence of equilibrium payoffs in stochastic games, or Nash threats or disagreement levels in bargaining theory. Zero-sum games present some nice properties that give hope for existence for results *à la* Blackwell: First, since a zero-sum game admits at most one Nash payoff, one avoids to compare the value

of sets of equilibrium payoffs. Second –as we shall see– the value of information is positive for those games.

A zero-sum game is described by a action set S_i for each player $i \in \{1,2\}$, and by a payoff function $g : S_1 \times S_2 \to \mathbb{R}$; $g(s_1, s_2)$ is the payoff to player 1, and the payoff to player 2 is $-g(s_1, s_2)$. By playing t_1 in S_1, player 1 guarantees $\inf_{s_2} g(t_1, s_2)$, thus player 1 can guarantee $v = \sup_{s_1} \inf_{s_2} g(s_1, s_2)$ by choosing appropriately t_1. Similarly, player 2 can guarantee that player 1 will gain no more than $w = \inf_{s_2} \sup_{s_1} g(s_1, s_2)$. The most remarkable result for zero-sum games is known as the minmax theorem, and asserts that under fairly general conditions, $v = w$. The corresponding value is then called the value of the game G, and we shall denote it $V(G)$. In particular, the min max theorem is true when S_1 and S_2 are mixed strategy sets over finite strategy sets. A simple consequence of the min max theorem is that all Nash payoffs must be equal to $V(G)$. In fact, since player 1 can guarantee $V(G)$ all Nash payoffs must be at least $V(G)$, and on the other side, player 2 can always force player 1 down to $V(G)$.

We now turn to the description of information structures. An information structure $\mathbf{E} = ((\Omega, \mathcal{A}, P), \mathcal{A}_1, \mathcal{A}_2, \kappa)$ is given by:

- A probability space of states of the world $(\Omega, \mathcal{A}, P)$,
- A sub-sigma algebra $\mathcal{A}_i \subseteq \mathcal{A}$ for each player i that describes i's information on ω.
- A mapping κ from $(\Omega, \mathcal{A})$ to a set K that describes the payoff relevant state.

Given an information structure $\mathbf{E} = ((\Omega, \mathcal{A}, P), \mathcal{A}_1, \mathcal{A}_2, \kappa)$, strategy sets S_1 and S_2 for 1 and 2, and a payoff function $g : S_1 \times S_2 \times K \to \mathbb{R}$ (payoffs now depend also on k), the Bayesian game $\Gamma(\mathbf{E}, G)$ is the extended game in which
1) ω is picked in Ω according to P
2) i is informed of the elements of $\mathcal{A}_i$ containing ω
3) i chooses $s_i \in S_i$
4) the payoff to 1 is $g(s_1, s_2, k)$

A strategy for i in $\Gamma(\mathbf{E}, g)$ is σ_i: $(\Omega, \mathcal{A}_i) \to S_i$

The corresponding expected payoff to player 1 is

$$\gamma(\sigma_1, \sigma_2) = \mathbf{E}_P g(\sigma_1(\omega), \sigma_2(\omega), \kappa(\omega))$$

The value of $\Gamma(\mathbf{E}, g)$ is denoted by $V(\mathbf{E}, g)$.

A simple relation between information structure stems from comparison in payoffs: we say that $\mathbf{E}$ is better than $\mathbf{F}$ for player 1 if for every game G, $V(\mathbf{E}, g) \geq V(\mathbf{F}, g)$. $\mathbf{E}$ is then preferable to $\mathbf{F}$ for player 1 in *every* zero-sum game. We note this relation $\mathbf{E} \geq_V \mathbf{F}$. The corresponding equivalence relation holds when $V(\mathbf{E}, g) = V(\mathbf{F}, g)$ for every g, we then write $\mathbf{E} \cong_V \mathbf{F}$. We address the question of how to characterize these relations in terms of the information of the players in $\mathbf{E}$ and $\mathbf{F}$.

We say that player 1's information is finer in $\mathbf{E}$ than in $\mathbf{F}$, and we note $\mathbf{E} \geq^1 \mathbf{F}$ when $\mathbf{E} = ((\Omega, \mathcal{A}, P), \mathcal{A}_1, \mathcal{A}_2, \kappa)$, $\mathbf{F} = ((\Omega, \mathcal{A}, P), \mathcal{B}_1, \mathcal{A}_2, \kappa)$, with $\mathcal{B}_1 \subseteq \mathcal{A}_1$. Similarly, we note $\mathbf{E} \geq^2 \mathbf{F}$ if $\mathbf{E} = ((\Omega, \mathcal{A}, P), \mathcal{A}_1, \mathcal{A}_2, \kappa)$, $\mathbf{F} = ((\Omega, \mathcal{A}, P), \mathcal{A}_1, \mathcal{B}_2, \kappa)$, with $\mathcal{A}_2 \subseteq \mathcal{B}_2$.

A simple property, that establishes that value of information is positive in zero-sum games, is the following:

Proposition 1 *Assume* $\mathbf{E} \geq^1 \mathbf{F}$ *or* $\mathbf{E} \geq^2 \mathbf{F}$*, then* $\mathbf{E} \geq_V \mathbf{F}$*.*

The proof of the above proposition is simple. Just notice that if $\mathbf{E} \geq^1 \mathbf{F}$, player 1 has more strategies in $\Gamma(\mathbf{E},g)$ than in $\Gamma(\mathbf{F},g)$, and that if $\mathbf{E} \geq^2 \mathbf{F}$, player 2 has less strategies in $\Gamma(\mathbf{E},g)$ than in $\Gamma(\mathbf{F},g)$. The min max is then increased either if the max is taken over a larger set, or if the min is taken over a smaller set.

We define a second way of comparing information structures with respect to information as follows. Let us write $\mathbf{E} \cong^1 \mathbf{F}$ whenever $\mathbf{E} = ((\Omega, \mathcal{A}, P), \mathcal{A}_1, \mathcal{A}_2, \kappa)$, $\mathbf{F} = ((\Omega, \mathcal{A}, P), \mathcal{B}_1, \mathcal{A}_2, \kappa)$, and $\mathcal{B}_1 \subseteq \mathcal{A}_1$ is a sufficient statistic for $\mathcal{A}_1$ on $(\mathcal{A}_2, k)$. In this case, $\mathcal{B}_1$ is as informative on $(\mathcal{A}_2, k)$ as $\mathcal{A}_1$. $\mathbf{E} \cong^2 \mathbf{F}$ is defined similarly.

According to the next proposition, sufficient statistics do not affect values.

Proposition 2 *Assume* $\mathbf{E} \cong^1 \mathbf{F}$ *or* $\mathbf{E} \cong^2 \mathbf{F}$*, then* $\mathbf{E} \cong_V \mathbf{F}$*.*

To prove this, assume $\mathbf{E} = ((\Omega, \mathcal{A}, P), \mathcal{A}_1, \mathcal{A}_2, \kappa)$, and $\mathcal{B}_1 \subseteq \mathcal{A}_1$ is a sufficient statistic for $\mathcal{A}_1$ on $(\mathcal{A}_2, k)$. We show that 2 cannot guarantee more in $\Gamma(\mathbf{E}, g)$ than in $\Gamma(\mathbf{F}, g)$. Indeed, for every strategy σ_2 of player 2 in $\Gamma(\mathbf{E}, g)$, player 1 has a best response in $\Gamma(\mathbf{E}, g)$ which is $\mathcal{B}_1$ measurable. This last property comes from the fact that for one-person decision problems, the optimal action depends on the belief on the state of nature only. Here, the relevant information to player 1 is the belief held on k and s_2, which depends (given σ_2) on the belief held on k and $\mathcal{A}_2$ only.

Using the previously defined relations, we denote $\mathbf{E} \geq_I \mathbf{F}$ when there exists a sequence $\mathbf{E}_0, \mathbf{E}_1, \ldots, \mathbf{E}_n$ such that $\mathbf{E}_0 = \mathbf{E}$, $\mathbf{E}_n = \mathbf{F}$, and for every k, one of the following holds: $\mathbf{E}_k \geq^1 \mathbf{E}_{k+1}$, $\mathbf{E}_k \geq^2 \mathbf{E}_{k+1}$, $\mathbf{E}_k \cong^1 \mathbf{E}_{k+1}$, $\mathbf{E}_{k+1} \cong^1 \mathbf{E}_k$, $\mathbf{E}_k \cong^2 \mathbf{E}_{k+1}$, or $\mathbf{E}_{k+1} \cong^2 \mathbf{E}_k$. We note $\mathbf{E} \cong_I \mathbf{F}$ when both $\mathbf{E} \geq_I \mathbf{F}$ and $\mathbf{F} \geq_I \mathbf{E}$. It comes as an immediate corollary of the above propositions that $\mathbf{E} \geq_I \mathbf{F}$ implies $\mathbf{E} \geq_V \mathbf{F}$. Our main result is the following:

Theorem 3 $\mathbf{E} \cong_V \mathbf{F}$ *if and only if* $\mathbf{E} \cong_I \mathbf{F}$*.*

We establish this by proving that, if $\mathbf{E} \cong_V \mathbf{F}$, then the relevant information to the players in $\mathbf{E}$ and in $\mathbf{F}$ is essentially the same. It remains to give a precise meaning to the previous statement. We present here very informally the ideas underlying the proof. First, we can establish the following lemma:

Lemma 4 *If* $\mathbf{E} \cong_V \mathbf{F}$*, then the information of player 1 on* k *in* $\mathbf{E}$ *and in* $\mathbf{F}$ *are the same.*

This can be proved using games in which the payoff depends on k and on player 1's action – in such games, player 2 is "dummy" and player 1 is asked to place a bet on the realized k. The result then comes from Blackwell's Theorem: both statistical experiments representing 1's information on k are equivalent in payoffs, therefore they are equivalent in information.

Then, one may prove:

Lemma 5 *If* $\mathbf{E} \cong_V \mathbf{F}$*, then the information of player 2 on the information of 1 on* k *in* $\mathbf{E}$ *and in* $\mathbf{F}$ *are the same.*

The idea of the proof is as this: Perturb the games in which player 1 bets on k by games in which player 2 bets on player 1's action. Player 1's strategies are little perturbed if the perturbation of payoffs is small, and player 2 having the same information on player 1's actions in $\mathbf{E}$ and in $\mathbf{F}$ must have the same information on 1's information on k.

Now, recall informally that the canonical information structure (Mertens and Zamir 86) associated to $\mathbf{E}$ is constituted of: player's information on k in $\mathbf{E}$, player's information on other player's information on k in $\mathbf{E}$, player's information on ... ... on player's information on k in $\mathbf{E}$, at all levels. A generalization of the logic of the above lemmata is: If $\mathbf{E} \cong_V \mathbf{F}$, player's information on k is the same, and can be extracted from player's actions in suitably designed games – we express this by saying that this information is *required* to play in those games.

Also, if a player's – say 1– information is required to play in some game, then there exists a game in which player 2's information on the required information of player 1 is also required. This way, we prove the existence of games in which all the canonical information is required. This in turn implies that if $\mathbf{E} \cong_V \mathbf{F}$, then $\mathbf{E}$ and $\mathbf{F}$ have the same canonical information structure associated.

To finish the proof of our theorem, it remains to see that if $\mathbf{E}$ and $\mathbf{F}$ have the same canonical information structure associated, then $\mathbf{E} \cong_I \mathbf{F}$. It is sufficient for this to prove that if $\mathbf{C}$ is the canonical information structure associated to $\mathbf{E}$, then $\mathbf{E} \cong_I \mathbf{C}$ – the relation $\cong_I$ being both reflexive and transitive. This can be done using properties of canonical information structures that can be found in chapter III of the book of Mertens, Sorin and Zamir (1994).

Our main result establishes the identity of equivalence classes for the two relations $\geq_I$ and $\geq_V$. This does not suffice to establish the equivalence of both relations and we conclude by an open question: do the two relations coincide? Namely, is it true that $\mathbf{E} \geq_V \mathbf{F}$ implies $\mathbf{E} \geq_I \mathbf{F}$?

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