MATHEMATICS OF OPERATIONS RESEARCH
Vol. 28, No. 1, February 2003, pp. 127–140
Printed in U.S.A.

ON THE COMPLEXITY OF COORDINATION

OLIVIER GOSSNER AND PENÉLOPE HERNÁNDEZ

Many results on repeated games played by finite automata rely on the complexity of the exact implementation of a coordinated play of length n. For a large proportion of sequences, this complexity appears to be no less than n. We study the complexity of a coordinated play when allowing for a few mismatches. We prove the existence of a constant C such that if $(m \ln m)/n \geq C$, for almost any sequence of length n, there exists an automaton of size m that achieves a coordination ratio close to 1 with it. Moreover, we show that one can take any constant C such that $C > e|X| \ln |X|$, where $|X|$ is the size of the alphabet from which the sequence is drawn. Our result contrasts with Neyman (1997) that shows that when $(m \ln m)/n$ is close to 0, for almost no sequence of length n there exists an automaton of size m that achieves a coordination ratio significantly larger $1/|X|$ with it.

1. Introduction. In the last two decades, models from computer science have been applied to game theory for the modelization of boundedly rational agents (see for instance Abreu and Rubinstein 1988, Anderlini 1989, Anderlini and Sabourian 1995, Aumann 1981, Ben-Porath 1993, Gossner 1998, 2000, Hernández and Urbano 2001a, 2001b, Kalai 1990, Kalai and Standford 1988, Lehrer 1988, 1994, Meggido and Widgerson 1989, Neyman 1985, 1997, 1998, Neyman and Okada 1999, 2000a, 2000b, Papadimitriou and Yannakakis 1994, 1998, Piccione and Rubinstein 2002, and Rubinstein 1986). In particular, Aumann (1981) proposed to measure the complexity of a strategy by the size of the smallest automaton implementing it. Repeated games played by finite automata have since then been studied by Neyman (1985, 1997, 1998) Rubinstein (1986), Kalai and Standford (1988), Abreu and Rubinstein (1988), and Ben Porath (1993) (see also the survey of Kalai). In these games, one is often led to measure the complexity of a play, defined as the least complexity of a strategy inducing this play (Kalai and Standford 1988, Neyman 1998).

A class of plays of particular interest consists of *coordinated plays*, in which there exists a one-to-one correspondence between the actions played by any pair of players. For most of them, this complexity is equal to the length of the play. This fact has been used to control the complexity of equilibrium paths by Abreu and Rubinstein (1988) and Neyman (1998), among others.

Coordinated plays also arise in two-player zero-sum games played by finite automata. From a one-shot zero-sum game, one defines a normal form game in which player 1 (resp. 2) chooses an automaton of size m (resp. n), and the payoff is the long-run payoff of the induced sequence of actions. To characterize the max min in pure and mixed strategies of these games, one has to study the ratio of coordination that the automaton of player 1 can achieve with the one of player 2.

In mixed strategies, for $m \geq \exp(Kn)$ (where K is the logarithm of the cardinality of the set of action of player 2), Neyman (1997) constructs a mixed strategy of player 1 that eventually enters into perfect coordination against all *strategies* of player 2 (and in particular against all n-periodic sequences), with probability close to 1. On the other direction, when m is subexponential in n ($\ln m \ll n$), Ben Porath (1993) exhibits a mixture of *n-periodic*

Received November 8, 2001; revised September 10, 2002.
MSC 2000 subject classification. Primary: 91A20.
OR/MS subject classification. Primary: Games/group decisions.
Key words. Coordination, complexity, automata.

0364-765X/03/2801/0127/$05.00

sequences against which no automaton can achieve a coordination ratio significantly larger than the minimal one.

To study the analogous in pure strategies, Neyman (1997) proved that when $m \ln m \ll n$, for almost any sequence of length n, there exists no automaton of size m that achieves significant coordination with this sequence.

Between perfect coordination and absence of coordination, we study the complexity of *almost perfect* coordination of an automaton with a periodic sequence. We prove the existence of a constant C such that if $(m \ln m)/n \geq C$, almost every n-periodic sequence can be almost perfectly predicted by an automaton of size m. Moreover, one can take any constant C such that $C > e|X| \ln |X|$, where X is the set in which the sequence takes its actions.

Our proof consists of a probabilistic and of a constructive part. We identify a subset of periodic sequences that verify a statistical regularity property, and call those sequences *regular*. Given a regular sequence, we construct an automaton that achieves almost perfect coordination with it. The probabilistic part consists in proving that almost all n-periodic sequences are regular.

We present the model and basic results in §2. Our main result is stated and proved in §3. We extend the result to almost sure convergence in §4. We conclude with some remarks in §5.

2. Preliminaries. For $z \in \mathbb{R}$, we let $\lfloor z \rfloor$ and $\lceil z \rceil$ denote the integer part and the superior integer part of z, respectively ($z - 1 < \lfloor z \rfloor \leq z$ and $z \leq \lceil z \rceil < z + 1$). Given a finite set Z, $|Z|$ denotes the cardinality of Z.

Let X be a finite set and let X_n represent the set of n-periodic sequences of elements of X.

A (*finite*) *automaton* $M \in FA(m)$ of size m with actions in X is a tuple $M = \langle Q, q^*, f, g \rangle$, where:

- Q is finite set of states, $|Q| = m$.
- $q^* \in Q$ is the initial state.
- $f\colon Q \to X$ is the action function.
- $g\colon Q \times X \to Q$ is the transition function.

An automaton $M = \langle Q, q^*, f, g \rangle \in FA(m)$ and a sequence $x = (x_t)_t$ induce a sequence of actions and states $(q^*, y_1, q_2, y_2, \ldots)$, where $y_1 = f(q^*)$, and for $t \geq 2$, $q_t = g(q_{t-1}, x_{t-1})$, $y_t = f(q_t)$. The corresponding sequence of actions $(y_t)_{t \geq 1}$ chosen by the automaton will be denoted $y(x, M)$. If $x^n \in X_n$, then $(x_t, y_t(x^n, M))_t$ is periodic of period at most mn after a finite number of stages.

The *ratio of occurrence* between a periodic sequence x^n and an automaton $M \in FA(m)$ is defined as

$$\rho(x^n, M) = \lim_{T \to \infty} \frac{1}{T} \big| \{1 \leq t \leq T \mid y_t(x^n, M) = x_t\} \big|.$$

$\rho(x^n, M)$ is the average proportion of stages for which M correctly predicts the sequence x^n. Given x^n, the best ratio of occurrences that an automaton of size m can achieve with x^n is

$$\rho^m(x^n) = \max_{M \in FA(m)} \rho(x^n, M).$$

The next lemma recalls some simple properties of $\rho^m(x^n)$.

LEMMA 1.
1. $1/|X| \leq \rho^m(x^n) \leq 1$.
2. *If* $m \geq n$, *then* $\rho^m(x^n) = 1$.
3. $x^n \in X_{n_0}$ *with* $n_0 = \inf_m \{\rho^m(x) = 1\}$.

PROOF. Obviously, $\rho^m(x^n) \leq 1$. To show that $\rho^m(x^n) \geq 1/|X|$, let $x \in X$ be an action occurring at least a proportion $1/|X|$ of the time in x^n and consider any automaton which plays x at every state. Point 2 is obvious, and Point 3 can be seen, for instance, as a consequence of Lemma 5 in Neyman (1998) that relates the complexity of a play with its periodicity. □

3. Asymptotic properties. We are concerned with asymptotic properties of the distribution of $\rho^m(x^n)$ when x^n is drawn uniformly in X_n. Let thus (x_i) be a sequence of i.i.d. random variables uniformly distributed in X, and $x^n \in X_n$ be the n-periodic sequence that coincides with (x_i) during its n first stages. Pr represents the induced probability on the sets X_n. We recall the following result from Neyman (1997):

THEOREM 2 (NEYMAN 1997). *For a sequence $(m(n))_n$ of positive integers, condition $\lim_{n\to\infty}((m(n)\ln m(n))/n) = 0$ implies*

$$\forall \varepsilon > 0, \quad \lim_{n\to\infty} \Pr\left(\rho^m(x^n) < \frac{1}{|X|} + \varepsilon\right) = 1.$$

This result provides an asymptotic condition on m and n, namely $(m \ln m)/n \to 0$, under which automata of size m cannot achieve coordination ratios larger than $1/|X|$ with almost any n-periodic sequence. Our main result states the existence of a constant C such that if $(m \ln m)/n$ is asymptotically larger than C, then automata of size n can achieve coordination ratios arbitrarily close to 1 with almost *all* sequences in X_n.

THEOREM 3. *There exists a constant C such that for any sequence of positive integers $(m(n))_{n\in\mathbb{N}}$ with $\lim_{n\to\infty}(m(n)\ln m(n))/n > C$,*

$$\forall \varepsilon, \quad \lim_{n\to\infty} \Pr(\rho^m(x^n) > 1 - \varepsilon) = 1.$$

In particular, one can take $C = e|X|\ln|X|$.

To prove this, we define in §3.1 a subset of X_n of sequences verifying a statistical regularity condition. We call those sequences *regular*. Then, in §3.2, for each regular sequence x^n, we construct an automaton in $FA(m)$ that achieves a large ratio of occurrences with x^n. Finally, in §§3.3 and 3.4, we prove that almost all n-periodic sequences are regular.

Figure 1 illustrates the different (m, n) regions and the corresponding coordination ratios when $|X| = 2$. Our result fills the gap between the regions $m \ln m \geq 2e\ln(2)n$ and $n \geq m$. No result is known in the region labeled "?".

3.1. Regularity. In this section, we define the statistical regularity condition that ensures a large ratio of occurrences. Let $x^n = (x_1, x_2, \ldots) \in X_n$ and $l \leq n$. We identify x^n to its n first elements, thus making the abuse of notation $x^n \in X^n$. For $1 \leq j < \lceil n/l \rceil$, we write $r_j = (x_{l(j-1)+1}, \ldots, x_{lj})$ and $r' = (x_{(\lceil n/l \rceil - 1)l}, \ldots, x_{n-1}, x_n)$. This way, x^n is seen as the concatenation of subsequences $r_1 \ldots r_{\lceil n/l \rceil - 1} r'$ with $r_j \in X^l$ for $1 \leq j < \lceil n/l \rceil$ and $r' \in X^{n - l(\lceil n/l \rceil - 1)}$. We complete r' into a subsequence of length l by setting $r_{\lceil n/l \rceil} = (0, \ldots, 0, x_{(\lceil n/l \rceil - 1)l+1}, \ldots, x_{n-1}, x_n) \in X^l$.

Given $r \in X^l$, the number of times that r appears in x^n is

$$S(x^n, r) = \left| \left\{ 0 \leq j \leq \left\lceil \frac{n}{l} \right\rceil \;\middle|\; \forall 1 \leq t \leq l, x_{jl+t} = r_t \right\} \right|.$$

We define the set of (k, l)-regular (or regular for short) sequences $R_l(n, k)$ as the subset of X_n such that for all $r \in X^l$, $S(x^n, r) \leq k$.

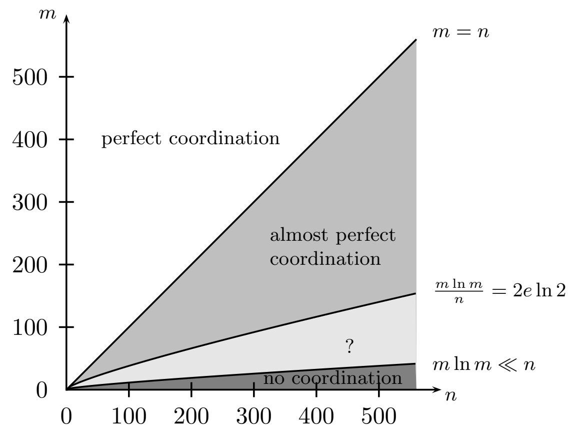

FIGURE 1. (m, n) regions and coordination.

3.2. Construction of an automaton for regular sequences.

PROPOSITION 4. *Assume $x^n \in R_l(n, k)$ and $m = k|X|^l$. Then*

$$\rho^m(x^n) \geq 1 - \frac{1}{n} - \frac{1}{l} \cdot$$

The proof of the proposition is constructive. We first present two examples to ilustrate the construction of the automaton, then turn to the general proof.

3.2.1. Example 1. Let $X = \{0, 1\}$, $n = 36$, and $m = 16$. We consider the following sequence $x \in X_{36}$:

$$x = 000011100010101110001100010101101111.$$

Regularity of x. We show that x is regular for $l = 4$ and $k = 1$. The sequence x writes as the concatenation $r_1 r_2 r_3 \ldots r_8 r_9$ of subsequences of length 4:

$$x = \underbrace{0000}_{r_1}\ \underbrace{1110}_{r_2}\ \underbrace{0010}_{r_3}\ \underbrace{1011}_{r_4}\ \underbrace{1000}_{r_5}\ \underbrace{1100}_{r_6}\ \underbrace{0101}_{r_7}\ \underbrace{0110}_{r_8}\ \underbrace{1111}_{r_9}.$$

All subsequences are distinct, thus $x \in R_4(36, 1)$. To illustrate Proposition 4, we construct $M = \langle Q, q^*, f, g\rangle \in FA(16)$ such that $\rho(M, x^{36}) = 3/4$.

Construction of the induced sequence of actions. For $r = (r^1, r^2, r^3, r^4) \in \{0, 1\}^4$, let $\bar{r} = (r^1, r^2, r^3, r^4 + 1 \bmod 2)$, so that r and $\bar{r}$ coincide except for their last elements.

We define $\bar{x} \in R_4(36, 1)$, $\bar{x} = \bar{r}_1\bar{r}_2\bar{r}_3\bar{r}_4\bar{r}_5\bar{r}_6\bar{r}_7\bar{r}_8\bar{r}_9$,

$$\bar{x} = \underbrace{0001}_{\bar{r}_1}\ \underbrace{1111}_{\bar{r}_2}\ \underbrace{0011}_{\bar{r}_3}\ \underbrace{1010}_{\bar{r}_4}\ \underbrace{1001}_{\bar{r}_5}\ \underbrace{1101}_{\bar{r}_6}\ \underbrace{0100}_{\bar{r}_7}\ \underbrace{0111}_{\bar{r}_8}\ \underbrace{1110}_{\bar{r}_9}.$$

We now design M such that the sequence of actions $y(x, M)$ is $\bar{x}^{36}$.

Construction of the state space and action function. Let $Q = \{1, \ldots, 16\}$ and $f\colon Q \to \{0, 1\}$ be defined by

$$f(1)f(2)\ldots f(16) = 0000111101011001.$$

The choice of the sequence $f(1)f(2)\dots f(16)$ is not arbitrary. It has the remarkable property that for every $r \in \{0, 1\}^4$ there exists a unique $i \in Q$ such that

$$r = (f(i \bmod 16), f(i+1 \bmod 16), f(i+2 \bmod 16), f(i+3 \bmod 16)).$$

This defines the following bijective map α_1 from $\{0, 1\}^4$ to Q:

$$\begin{aligned} \{0,1\}^4 &\to Q \\ 0000 &\mapsto 1 \\ 0001 &\mapsto 2 \\ 0011 &\mapsto 3 \\ 0111 &\mapsto 4 \\ 1111 &\mapsto 5 \\ 1110 &\mapsto 6 \\ 1101 &\mapsto 7 \\ 1010 &\mapsto 8 \\ 0101 &\mapsto 9 \\ 1011 &\mapsto 10 \\ 0110 &\mapsto 11 \\ 1100 &\mapsto 12 \\ 1001 &\mapsto 13 \\ 0010 &\mapsto 14 \\ 0100 &\mapsto 15 \\ 1000 &\mapsto 16 \end{aligned}$$

Because $t \neq t'$ implies $r_t \neq r_{t'}$, there exists an injective α: $\{1, \dots, 9\} \to Q$ such that

$$r_t = (f(\alpha(t) \bmod 16), f(\alpha(t)+1 \bmod 16), f(\alpha(t)+2 \bmod 16), f(\alpha(t)+3 \bmod 16)).$$

More precisely

$$\begin{aligned} \alpha\colon \{1,\dots,9\} &\to Q \\ 1 &\mapsto 2 \\ 2 &\mapsto 5 \\ 3 &\mapsto 3 \\ 4 &\mapsto 8 \\ 5 &\mapsto 13 \\ 6 &\mapsto 7 \\ 7 &\mapsto 15 \\ 8 &\mapsto 4 \\ 9 &\mapsto 6 \end{aligned}$$

Construction of the transition function and initial state. We let $q^* = \alpha(1) = 2$ be the initial state. We construct the transition function g in such a way that when M predicts correctly the element of x^n, it goes to the next state:

$$g(q, f(q)) = q + 1 \bmod 16.$$

We now define the transition function when M does not predict the correct element of x^n.

- Assume $q = \alpha(t) + 3$ for some t. This t must be unique, for α is injective. We then let $g(q, f(q) + 1 \bmod 2) = \alpha(t + 1 \bmod 9)$.
- If there exists no t such that $q = \alpha(t) + 3$, we let $g(q, f(q) \bmod 2)$ be arbitrary.

Figure 2 represents the states and transitions of M. The states are $1, 2, \dots, 16$. The solid arrows ($\longrightarrow$) represent the transitions when M predicts the correct action, and the dotted arrows ($\cdots>$) correspond to the transitions of M after a mistake. The states of the automaton when no mistake occurs follow a cycle of size $|Q| = 16$.

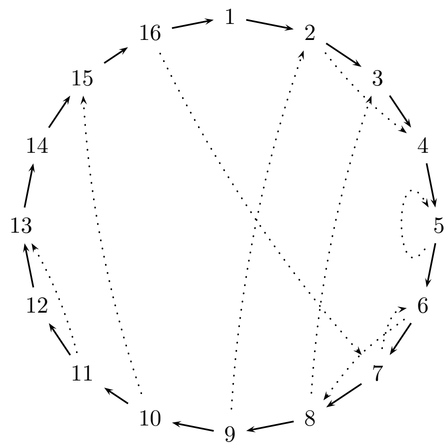

FIGURE 2. States and transitions for Example 1.

The induced sequence of actions and states. Let $(q^*, y_1, q_2, y_2, \ldots)$ be the sequence of actions and states induced by M and x^{36}. We establish by induction on $t \geq 0$ that $q_{4t+1} = \alpha(t+1)$. The property is verified for $t = 0$ by definition of q^*. If $q_{4t+1} = \alpha(t+1)$, then we can follow the actions and states at stages $4t+1, \ldots, 4(t+1)$.

$$\begin{aligned} y_{4t+1} &= \bar{x}_{4t+1} = x_{4t+1} & q_{4t+2} &= \alpha(t+1)+1 \\ y_{4t+2} &= \bar{x}_{4t+2} = x_{4t+2} & q_{4t+3} &= \alpha(t+1)+2 \\ y_{4t+3} &= \bar{x}_{4t+3} = x_{4t+3} & q_{4(t+1)} &= \alpha(t+1)+3 \\ y_{4(t+1)} &= \bar{x}_{4(t+1)} \neq x_{4(t+1)} & q_{4(t+1)+1} &= \alpha(t+2) \end{aligned}$$

By definition of α, the actions played at stages $4t+1, \ldots, 4(t+1)$ follow $\bar{r}_t = (\bar{x}_{4t+1}, \bar{x}_{4t+2}, \bar{x}_{4t+3}, \bar{x}_{4(t+1)})$. Therefore, the first mistake occurs at stage $4(t+1)$. The corresponding state is $q_{4(t+1)} = \alpha(t)+3$, so that the transition maps to $q_{4(t+1)+1} = \alpha(t+1)$.

The following table summarizes the subsequences r_t, $\bar{r}_t$, and the states $q_{4(t-1)+1}$, q_{4t} for $t \in \{1 \ldots 9\}$.

r_t	0000	1110	0010	1011	1000	1100	0101	0110	1111
$\bar{r}_t$	0001	1111	0011	1010	1001	1101	0100	0111	1101
$q_{4(t-1)+1}$	2	5	3	8	13	7	15	4	6
q_{4t}	5	8	6	11	16	10	2	7	9

At stage 37, the state is $q_{4\times 9+1} = \alpha(1) = q^*$. The sequence of actions and states is then periodic of period 36. Furthermore, $\forall t \geq 0$, $y_t = \bar{x}_t$, which is the required property. Thus the ratio of occurrences between x and M is 3/4, which of course implies that $\rho^{16}(x) \geq 3/4$.

3.2.2. Example 2. We now illustrate the construction of the automaton when the sequence is regular for $k = 2$. Here, the set of states consists of two identical "copies" of the states of the previous example. We follow the same procedure as in Example 1.

Let $X = \{0, 1\}$, $n = 72$, $m = 32$, and $x \in X_{72}$:

$$\begin{aligned} x = {} & 000011100000000101011100011100010110 0 \\ & 101101010110010101101000111111001110. \end{aligned}$$

Regularity of x. The sequence x writes as the concatenation $r_1 r_2 r_3 \dots r_{17} r_{18}$ of subsequences of length 4:

$$x = \underbrace{0000}_{r_1}\ \underbrace{1110}_{r_2}\ \underbrace{0000}_{r_3}\ \underbrace{0010}_{r_4}\ \underbrace{1011}_{r_5}\ \underbrace{1000}_{r_6}\ \underbrace{1110}_{r_7}\ \underbrace{0010}_{r_8}\ \underbrace{1100}_{r_9}$$
$$\underbrace{1011}_{r_{10}}\ \underbrace{0101}_{r_{11}}\ \underbrace{0110}_{r_{12}}\ \underbrace{0101}_{r_{13}}\ \underbrace{0110}_{r_{14}}\ \underbrace{1000}_{r_{15}}\ \underbrace{1111}_{r_{16}}\ \underbrace{1100}_{r_{17}}\ \underbrace{1110}_{r_{18}}.$$

All subsequences appear at most twice, thus $x \in R_4(72, 2)$.

Construction of the induced sequence of actions. Let $\bar{x} = \bar{r}_1\bar{r}_2\bar{r}_3\bar{r}_4\bar{r}_5\bar{r}_6\bar{r}_7\bar{r}_8\bar{r}_9\bar{r}_{10}\bar{r}_{11}\bar{r}_{12}\bar{r}_{13}\bar{r}_{14}\bar{r}_{15}\bar{r}_{16}\bar{r}_{17}\bar{r}_{18} \in R_4(72, 2)$.

$$\bar{x} = \underbrace{0001}_{\bar{r}_1}\ \underbrace{1111}_{\bar{r}_2}\ \underbrace{0001}_{\bar{r}_3}\ \underbrace{0011}_{\bar{r}_4}\ \underbrace{1010}_{\bar{r}_5}\ \underbrace{1001}_{\bar{r}_6}\ \underbrace{1111}_{\bar{r}_7}\ \underbrace{0011}_{\bar{r}_8}\ \underbrace{1101}_{\bar{r}_9}$$
$$\underbrace{1010}_{\bar{r}_{10}}\ \underbrace{0100}_{\bar{r}_{11}}\ \underbrace{0111}_{\bar{r}_{12}}\ \underbrace{0100}_{\bar{r}_{13}}\ \underbrace{0111}_{\bar{r}_{14}}\ \underbrace{1001}_{\bar{r}_{15}}\ \underbrace{1110}_{\bar{r}_{16}}\ \underbrace{1101}_{\bar{r}_{17}}\ \underbrace{1111}_{\bar{r}_{18}}.$$

We now design M such that the sequence of actions $y(x, M)$ is $\bar{x}^{72}$.

Construction of the state space and action function. The states of M are formed by two copies of the states of Example 1. Let then $Q = \{1, \dots, 16\} \times \{1, 2\}$, and $f\colon Q \to \{0, 1\}$ be given by

$$f(1, j)f(2, j)\dots f(16, j) = 0000111101011001, \qquad j \in \{1, 2\}.$$

For every $\bar{r} \in \bar{x}$, there exists a unique $i \in \{1 \dots 16\}$ such that for $j = 1, 2$:

$$\bar{r} = (f(i \bmod 16, j), f(i+1 \bmod 16, j), f(i+2 \bmod 16, j), f(i+3 \bmod 16, j)).$$

Because $x \in R_4(72, 2)$, there exists an injective map $\alpha\colon \{1 \dots 18\} \to Q$ such that $\alpha(t) = (i, j)$ implies

$$\bar{r}_t = ((f(i \bmod 16), j), (f(i+1 \bmod 16), j), (f(i+2 \bmod 16), j), (f(i+3 \bmod 16), j)).$$

For instance, take

$$\begin{aligned}
\alpha\colon \{1, \dots, 18\} &\to Q\\
1 &\mapsto (2, 1)\\
2 &\mapsto (5, 1)\\
3 &\mapsto (2, 2)\\
4 &\mapsto (3, 2)\\
5 &\mapsto (8, 2)\\
6 &\mapsto (13, 2)\\
7 &\mapsto (5, 2)\\
8 &\mapsto (3, 1)\\
9 &\mapsto (7, 1)\\
10 &\mapsto (8, 1)\\
11 &\mapsto (15, 1)\\
12 &\mapsto (4, 1)\\
13 &\mapsto (15, 2)\\
14 &\mapsto (4, 2)\\
15 &\mapsto (13, 1)\\
16 &\mapsto (6, 1)\\
17 &\mapsto (7, 2)\\
18 &\mapsto (6, 2)
\end{aligned}$$

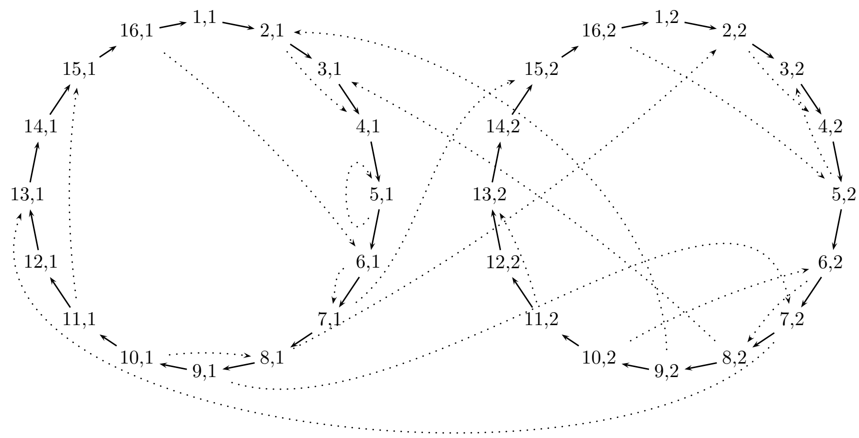

FIGURE 3. States and transitions for Example 2.

Notice that $r_1 = r_3$, $r_2 = r_7$, $r_4 = r_8$, $r_5 = r_{10}$, $r_6 = r_{15}$, $r_9 = r_{17}$, $r_{11} = r_{13}$, $r_{12} = r_{14}$, and $r_{16} = r_{18}$. All those "pairs" of subsequences are mapped to the same i in different j's.

Construction of the transition function and initial state. We let $q^* = \alpha(1) = (2, 1)$ be the initial state.

When predicting the correct element of the sequence, M goes to the next i with the same j:

$$g((i, j), f(i, j)) = (i + 1 \bmod 16, j).$$

When M does not predict the correct element of x^n, the transition function follows from α and is defined by

$$g(\alpha(t) + (3, 0), f(\alpha(t) + (3, 0)) + 1 \bmod 2) = \alpha(t + 1 \bmod 18).$$

Figure 3 represents the states and transitions of M using the same notations as in Example 1.

The table below sums up the subsequences r_t and $\bar{r}_r$ that constitute x and $\bar{x}$ and the corresponding states $q_{4(t-1)+1}$ and q_{4t} of M when starting and finishing playing $\bar{r}_t$.

r_t	0000	1110	0000	0010	1011	1000	1110	0010	1100
$\bar{r}_t$	0001	1111	0001	0011	1010	1001	1111	0011	1101
$q_{4(t-1)+1}$	(2,1)	(5,1)	(2,2)	(3,2)	(8,2)	(13,2)	(5,2)	(3,1)	(7,1)
q_{4t}	(5,1)	(8,1)	(5,2)	(6,2)	(11,2)	(16,2)	(8,2)	(6,1)	(10,1)
r_t	1011	0101	0110	0101	0110	1000	1111	1100	1111
$\bar{r}_t$	1010	0100	0111	0100	0111	1001	1110	1101	1110
q_{4t+1}	(8,1)	(15,1)	(4,1)	(15,2)	(4,2)	(13,1)	(6,1)	(7,2)	(6,2)
q_{4t+4}	(11,1)	(2,1)	(7,1)	(2,2)	(7,2)	(16,1)	(9,1)	(10,2)	(9,2)

It is verified as in Example 1 that $y(M, x)$ is 72-periodic and coincides with $\bar{x}$ for the 72 first stages, hence $\rho(M, x) = 3/4$.

3.2.3. Proof of Proposition 4. We assume without loss of generality that $X = \{0, \ldots, |X| - 1\}$. For simplicity we now write x for x^n. We present the construction of an automaton $M = \langle Q, q^*, f, g\rangle \in FA(m)$ that ensures a high coincidence ratio with $x \in R_l(n, k)$. We start by the definition of the sequence of actions induced by M and x. Second, we design Q and f. Then we define g and q^*. Finally, we check that the induced sequence of actions is the desired one.

Construction of the induced sequence of actions. Recall that $x = r_1 \dots r_{\lceil n/l \rceil -1} r'$, with $r_t \in X^l$ for $1 \le t \le \lceil n/l \rceil - 1$ and that r' is completed by adjunction of 0's to the left into a subsequence $r_{\lceil n/l \rceil}$ of length l. For $r = (r^1, \dots, r^l) \in X^l$, let $\bar{r} = (r^1, \dots, r^{l-1}, r^l + 1 \bmod |X|)$.

Let $\bar{r}' = (x_{(\lceil n/l \rceil -1)l+1}, \dots, x_{n-1}, x_n + 1 \bmod |X|)$, and $\bar{x} = \bar{r}_1 \dots \bar{r}_{\lceil n/l \rceil -1} \bar{r}'$. For $1 \le t \le \lceil n/l \rceil$, $\bar{r}_t$ and r_t coincide except for their terminal elements. Thus, x and $\bar{x}$ coincide except at stages l, $2l$, $\dots$, $(\lceil n/l \rceil - 1)l$, and n.

Lemma 5. *If $x \in R_l(n, k)$, then $\bar{x} \in R_l(n, k)$.*

Proof. There is a bijective correspondence between r and $\bar{r}$ in X^l, and $S(\bar{x}, \bar{r}) = S(x, r)$. □

Construction of the state space and action function. The state space and action function we design depend only on k and l; they are independent of n and of the particular element x of $R_l(n, k)$. Our construction relies on a sequence of elements of X of minimal length in which each subsequence of length l appears (only) once, the existence of which is ensured by the following lemma.

Lemma 6. *Let $l \in \mathbb{Z}^+$. There exists a sequence $s = (s_1, \dots, s_{|X|^l})$ of elements of X such that for every $r \in X^l$, there exists $i \in \{1, \dots, |X|^l\}$ with*

$$r = (s_{i \bmod |X|^l}, \dots, s_{i+l \bmod |X|^l}).$$

Note that because the length of the sequence equals the size of X^l, i is necessarily unique for each r. The existence of such sequences, called DeBrujn sequences, is well known in computer science and can be proved using elementary graph theory (cf. for instance, van Lint and Wilson 2001, Chapter 8, p. 56). Another application of DeBrujn sequences to bounded rationality is due to Piccione and Rubinstein (2002).

Let $Q = \{1, \dots, |X|^l\} \times \{1, \dots, k\}$ be the set of states. For a state $q = (i, j)$ and $c \in \mathbb{N}$ we let $q + c = (i + c \bmod |X|^l, j)$.

We fix a sequence $s = (s_1, \dots, s_{|X|^l}) \in X^{|X|^l}$ as in Lemma 6, and define the action function f by $f(i, j) = s_i$.

Construction of the transition function and initial state. The crucial element of the construction is the existence of a map between the index of the subsequences r_t to Q, as stated by the following lemma.

Lemma 7. *There exists an injective map α from $\{1, \dots \lceil n/l \rceil\}$ to Q such that for $1 \le t \le \lceil n/l \rceil$, $(f(\alpha(t)), \dots, f(\alpha(t) + l)) = \bar{r}_t$.*

Proof. By construction of Q and f

$$\forall r \in X^l \quad \big|\{q \mid (f(q), \dots, f(q+l)) = r\}\big| = k.$$

On the other hand, because $\bar{x} \in R_l(n, k)$, by Lemma 5

$$\forall r \in X^l \quad \big|\{t \mid \bar{r}_t = r\}\big| \le k.$$

Hence the result. □

Let the initial state be $q^* = \alpha(1)$.

We define the transition function g in such a way that M goes from q to $q + 1$ when it predicts the right action:

$$g(q, f(q)) = q + 1.$$

We now define $g(q, a)$ for $a \ne f(q)$.

- If $q = \alpha(t) + l - 1$ for some $1 \le t \le \lceil n/l \rceil$, this t is then unique because α is injective.
  — If $t \neq \lceil n/l \rceil - 1$, let $g(q, a) = \alpha(t + 1 \bmod \lceil n/l \rceil)$ for all $a \neq f(q)$. -
  — If $t = \lceil n/l \rceil - 1$, let $g(q, a) = \alpha(\lceil n/l \rceil) + l\lceil n/l \rceil - n$ for all $a \neq f(q)$.
- If there exists no t such that $q = \alpha(t) + l - 1$ we let $g(q, a)$ when $a \neq f(q)$ arbitrary.

The special definition for $t = \lceil n/l \rceil - 1$ comes from the fact than the last subsequence of actions in $\bar{x}$ is not $\bar{r}_{\lceil n/l \rceil}$ but rather its last part $\bar{r}'$.

The induced sequence of actions and states. We now check that the induced sequence of actions is $\bar{x}$.

LEMMA 8. $y(M, x) = \bar{x}$.

PROOF. Let $(q^*, y_1, q_2, \dots)$ be the sequence of states and actions induced by M and x. We prove by induction that for $t = 0, \dots, \lceil n/l \rceil$, $q_{lt+1} = \alpha(t+1)$. This property is verified for $t = 0$ because $q^* = \alpha(r_1)$. Assume it is true for some $t < \lceil n/l \rceil$. From the definition of α, the sequence of actions played by M coincide with r_t at stages $lt + 1, \dots, l(t+1) - 1$ and differ at stage $l(t+1)$:

$$
\begin{array}{lll}
y_{lt+1} = \bar{x}_{lt+1} = x_{lt+1} & & q_{lt+2} = \alpha(t+1) + 1 \\
y_{lt+2} = \bar{x}_{lt+2} = x_{lt+2} & & q_{lt+3} = \alpha(t+1) + 2 \\
\vdots \quad \vdots \quad \vdots & & \vdots \quad \vdots \\
y_{l(t+1)-1} = \bar{x}_{l(t+1)-1} = x_{l(t+1)-1} & & q_{l(t+1)} = \alpha(t+1) + l - 1 \\
y_{l(t+1)} = \bar{x}_{l(t+1)} \neq x_{l(t+1)} & & q_{l(t+1)+1} = \alpha(t+2)
\end{array}
$$

So, the property is established by induction on t. Furthermore, we have proved that $(y_{lt+1}, \dots, y_{(l+1)t}) = \bar{r}_t$ for those t. The sequence of actions and states from stage $l(\lceil n/l \rceil - 1) + 1$ to n is then

$$
\begin{array}{lll}
y_{l(\lceil n/l \rceil-1)+1} = \bar{x}_{l(\lceil n/l \rceil-1)+1} = x_{l(\lceil n/l \rceil-1)+1} & & q_{l(\lceil n/l \rceil-1)+2} = \alpha(\lceil n/l \rceil) + l\lceil n/l \rceil - n + 1 \\
y_{l(\lceil n/l \rceil-1)+2} = \bar{x}_{l(\lceil n/l \rceil-1)+2} = x_{l(\lceil n/l \rceil-1)+2} & & q_{l(\lceil n/l \rceil-1)+3} = \alpha(\lceil n/l \rceil) + l\left\lceil \frac{n}{l} \right\rceil - n + 2 \\
\vdots \quad \vdots \quad \vdots & & \vdots \quad \vdots \\
y_{n-1} = \bar{x}_{n-1} = x_{n-1} & & q_n = \alpha(\lceil n/l \rceil) + l - 1 \\
y_n = \bar{x}_n \neq x_n & & q_{n+1} = \alpha(1)
\end{array}
$$

We have thus proved that $y(M, x)$ and $\bar{x}$ coincide during the first n stages and that $q_{n+1} = q^*$. This implies that $y(M, x)$ is n-periodic and that $y(M, x) = \bar{x}$. $\square$

PROOF OF PROPOSITION 4. The automaton M constructed is such that $y(M, x)$ is n-periodic and coincides with x^n at all first n stages except l, $2l$, $\dots$, $(\lceil n/l \rceil - 1)l$, and n. The ratio of occurrences is therefore

$$\rho(x, M) = 1 - \frac{\lceil n/l \rceil}{n} \ge 1 - \frac{1}{n} - \frac{1}{l}. \quad \square$$

3.3. Probability of regular sequences. In this section, we estimate the probability of the set of regular sequences $R_l(n, k)$. Recall that Pr is the probability on the set of n-periodic sequences induced by n i.i.d. uniformly distributed random variables in X.

LEMMA 9. *Let n, l, k in $\mathbb{N}^*$. Then*

$$\Pr(R_l(n, k)) \ge 1 - \frac{|X|^l}{(k+1)!} \left(\frac{\lceil n/l \rceil}{|X|^l} \right)^{k+1}.$$

PROOF. For a subset I of $k+1$ elements of $\{1, \ldots, \lceil n/l \rceil\}$, let E_I be the event $E_I = \{\forall t, t' \in I,\ r_t = r_{t'}\}$. Then

$$\Pr(E_I) = \left(\frac{1}{|X|^l}\right)^k$$

and

$$\begin{aligned}\Pr(R_l(n,k)) &\geq 1 - \sum_I \Pr(E_I)\\ &\geq 1 - \binom{\lceil n/l \rceil}{k+1}\left(\frac{1}{|X|^l}\right)^k\\ &\geq 1 - \frac{\lceil n/l \rceil^{k+1}}{(k+1)!}\left(\frac{1}{|X|^l}\right)^k\\ &\geq 1 - \frac{|X|^l}{(k+1)!}\left(\frac{\lceil n/l \rceil}{|X|^l}\right)^{k+1}. \quad \square\end{aligned}$$

Under some conditions on sequences $k(n)$ and $l(n)$, we provide an asymptotic bound on the probability of nonregular sequences:

LEMMA 10. *Let* $k, l : \mathbb{N} \to \mathbb{N}$ *such that* $\lim_{n\to\infty} k(n) = \lim_{n\to\infty} l(n) = +\infty$ *and*

$$\frac{k(n)l(n)|X|^{l(n)}}{n} \geq \gamma.$$

Then

$$1 - \Pr(R_{l(n)}(n, k(n))) = |X|^{l(n)}\left(\frac{e}{\gamma}\right)^{k(n)} O(1)$$

when $n \to \infty$.

PROOF. Denote k and l for $k(n)$ and $l(n)$. Using Lemma 9 and Stirling's formula we obtain

$$1 - \Pr(R_l(n,k)) = \frac{|X|^l}{\sqrt{k}}\left(\frac{\lceil n/l \rceil\, e}{(k+1)|X|^l}\right)^{k+1} O(1).$$

Remark that $kl/n \to_{n\to\infty} 0$ implies

$$\left(\frac{n/l+1}{n/l}\right)^{k+1} = (1 + l/n)^{k+1} \to 1$$

so that

$$\left(\frac{\lceil n/l \rceil}{n/l}\right)^{k+1} \to 1$$

and

$$\begin{aligned}1 - \Pr(R_l(n,k)) &= \frac{|X|^l}{\sqrt{k}}\left(\frac{ne}{l(k+1)|X|^l}\right)^{k+1} O(1)\\ &= \frac{|X|^l}{\sqrt{k}}\left(\frac{e}{\gamma}\right)^{k+1} O(1). \quad \square\end{aligned}$$

3.4. Proof of Theorem 3. Consider a sequence $m(n)$ such that

$$\lim_{n\to\infty} \frac{m(n)\ln m(n)}{n} > e|X|\ln|X|.$$

We exhibit sequences $l(m) \to \infty$ and $k(n) \to \infty$ such that $m(n) \geq k(n)|X|^{l(n)}$ and $\Pr(R_{l(n)}(n, k(n))) \to 1$. Theorem 3 is then a consequence of Proposition 4. We denote $m(n)$ by m, and similarly for k, l.

First, fix α such that

$$e < \alpha < \frac{1}{|X| \ln |X|} \lim_{n\to\infty} \frac{m \ln m}{n}$$

and $\beta > 0$ such that $|X|^\beta e < \alpha$. The map $x \mapsto (\beta/\alpha)x^2|X|^{\beta x}$ is continuous, strictly increasing on $\mathbb{R}^+$, takes the value 0 at $x = 0$, and tends to $+\infty$ when $x \to \infty$. Let thus k_0 be the unique positive solution of

$$n = \frac{\beta}{\alpha} k_0^2 |X|^{\beta k_0}$$

and $l_0 = \beta k_0$. Finally, let $k = [k_0]$ and $l = \lceil l_0 \rceil$.

LEMMA 11. $\lim_{n\to\infty} \Pr(R_l(n, k)) = 1$.

PROOF. Clearly, $\lim_{n\to\infty} l = \lim_{n\to\infty} k = \infty$. We also have

$$\frac{kl|X|^l}{n} \geq \frac{k}{k_0} \frac{k_0 l_0 |X|^{l_0}}{n} = \frac{k}{k_0}\alpha.$$

We can thus apply Lemma 10 with $|X|^\beta e < \gamma < \alpha$ and obtain

$$\begin{aligned} 1 - \Pr(R_l(n, k)) &= |X|^l \left(\frac{e}{\gamma}\right)^k O(1) \\ &= |X|^{l_0} \left(\frac{e}{\gamma}\right)^{k_0} O(1) \\ &= \left(\frac{|X|^\beta e}{\gamma}\right)^{k_0} O(1). \quad \square \end{aligned}$$

LEMMA 12. *For n large enough, $m \geq k|X|^l$.*

PROOF. With $m' = k|X|^l$,

$$\begin{aligned} \limsup \frac{m' \ln m'}{n} &= \limsup \frac{k|X|^l l \ln |X|}{(\beta/\alpha) k_0^2 |X|^{l_0}} \\ &\leq \alpha |X| \ln |X| \\ &< \liminf \frac{m \ln m}{n}. \end{aligned}$$

Hence the result. $\square$

4. Almost sure convergence. Any infinite sequence $x = (x_i)_i$ induces a sequence $(x^n)_n$, $x^n \in X_n$, such that x^n and x^{n+1} coincide for their n first stages. We now view Pr as a probability on those $(x^n)_n$. Theorem 3 presents a condition under which $\rho^m(x^n)$ converges to 1 in probability. We strengthen this result with convergence almost surely.

THEOREM 13. *There exists a constant C such that for any sequence of positive integers $(m(n))_{n\in\mathbb{N}}$ with $\lim_{n\to\infty}(m(n) \ln m(n))/n > C$,*

$$\lim_{n\to\infty} \rho^{m(n)}(x^n) = 1 \qquad \Pr \textit{ almost surely.}$$

In particular, one can take $C = e|X| \ln |X|$.

PROOF. We simply adapt the proof of Theorem 3. Let again α, γ be such that

$$e < \gamma < \alpha < \frac{1}{|X| \ln |X|} \lim_{n \to \infty} \frac{m \ln m}{n}$$

and now let $\beta > 0$ be such that $|X|^{2\beta} e < \gamma$. Let $k_0 \in \mathbb{R}^+$ be the solution of $n = (\beta/\alpha) k_0^2 |X|^{\beta k_0}$, $l_0 = \beta k_0$, $k = [k_0]$, and $l = \lceil l_0 \rceil$. It is seen as in the proof of Lemma 12 that $m \geq k|X|^l$ for n large enough. Let $y = |X|^\beta e/\gamma$. Applying Lemma 10,

$$1 - \Pr(R_l(n, k)) = y^{k_0} O(1).$$

Substituting k_0 with $\ln(\alpha n/\beta k_0^2)(1/(\beta \ln |X|))$ yields

$$\begin{aligned} 1 - \Pr(R_l(n, k)) &= \exp\left(\ln\left(\frac{\alpha n}{\beta k_0^2}\right) \frac{\ln y}{\beta \ln |X|}\right) O(1) \\ &= \left(\frac{\alpha n}{\beta k_0^2}\right)^{(\ln y)/(\beta \ln |X|)} O(1). \end{aligned}$$

Because k_0 is negligible compared to any positive power of n and because $\ln y/(\beta \ln |X|) < -1$, there exists $\varepsilon > 0$ such that $1 - \Pr(R_l(n, k)) = o(n^{-1-\varepsilon})$. From the first Borel-Cantelli lemma, the set $\{n,\ x^n \notin R_l(n, k)\}$ is finite for almost all $x = (x_i)_i$, hence the almost sure convergence of $\rho^m(x^n)$ to 1. □

5. Remarks and open problems. Proposition 2 in Neyman (1997) states that for any finite zero-sum game $G = (A, B, g)$, if $m \log m \ll n$, there exists a n-periodic sequence of actions in A that guarantees the value of G against all automata of size m with actions in B. Neyman actually proves that for any $\mu \in \Delta(A)$, almost any n-periodic sequence randomly chosen according to $\mu^{\otimes n}$ is essentially undistiguishable from an infinite sequence of i.i.d. random variables of distribution μ by automata of size m.

We have proved that if $m \log m > Cn$, then almost any *uniformly chosen* n-periodic sequence is almost fully predictable by some automaton of size m. We do not know if such a result holds when the choice of the random choice is with respect to $\mu^{\otimes n}$ for a general $\mu \in \Delta(A)$. Another question that we leave open is the one of the size m with respect to n such that *all* n-periodic sequences are almost fully predictible by automata of size m.

Acknowledgments. The authors thank Françoise Forges and Bernhard von Stengel for helpful conversations and Abraham Neyman for enlightening comments. Remarks by two anonymous referees and the associate editor are also gratefully acknowledged. The second author gratefully acknowledges financial support from the European Community programme under Contract Number HPMF-CT-2001-01451.

References

Abreu, D., A. Rubinstein. 1988. The structure of Nash equilibrium in repeated games with finite automata. *Econometrica* **56** 1259–1281.

Anderlini, L. 1989. Some notes on Church's thesis and common interest games. *Theory and Decisions* **29** 19–52.

———, H. Sabourian. 1995. Cooperation and effective computability. *Econometrica* **63** 1337–1369.

Aumann, R. J. 1981. *Survey of Repeated Games*. Wissenschaftsverlag, Bibliographisches Institut, Mannheim, Wien, Zurich, 11–42.

Ben-Porath, E. 1993. Repeated games with finite automata. *J. Econom. Theory* **59** 17–32.

Gossner, O. 1998. Repeated games played by cryptographically sophisticated players. *CORE DP* 9835, CORE, Louvain-la-Neuve, Belgique.

———. 2000. Sharing a long secret in a few public words. *DP* 2000-15, THEMA, Nanterre, France.

Hernández, P., A. Urbano. 2001a. Communication and automata. *WP-AD* 2001-04, IVIE, Valencia, Spain.

———, ———. 2001b. Pseudorandom processes: Entropy and automata. *WP-AD* 2001-22, IVIE, Valencia, Spain.

Kalai, E. 1990. Bounded rationality and strategic complexity in repeated games. T. Ichiishi, A. Neyman, Y. Tauman, eds. *Game Theory and Applications*. Academic Press, New York, 131–157.

———, W. Standford. 1988. Finite rationality and interpersonal complexity in repeated games. *Econometrica* **56**(2) 397–410.

Lehrer, E. 1988. Repeated games with stationary bounded recall strategies. *J. Econom. Theory* **46** 130–144.

———. 1994. Finitely many players with bounded recall in infinitely repeated games. *Games and Econom. Behavior* **7** 390–405.

Meggido, N., R. Widgerson. 1989. On computable beliefs of rational machines. *Games and Econom. Behavior* **1** 144–169.

Neyman, A. 1985. Bounded complexity justifies cooperation in the finitely repeated prisoner's dilemma. *Econom. Lett.* **19** 227–229.

———. 1997. Cooperation, repetition, and automata. S. Hart, A. Mas Colell, eds. *Cooperation: Game-Theoretic Approaches*. NATO ASI Series F, 155. Springer-Verlag, Berlin-Heidelberg-New York, 233–255.

———. 1998. Finitely repeated games with finite automata. *Math. Oper. Res.* **23** 513–552.

———, D. Okada. 1999. Strategic entropy and complexity in repeated games. *Games and Econom. Behavior* **29** 191–223.

———, ———. 2000a. Repeated games with bounded entropy. *Games and Econom. Behavior* **30** 228–247.

———, ———. 2000b. Two-person repeated games with finite automata, *Internat. J. Game Theory* **29** 309–325.

Papadimitriou, C., M. Yannakakis. 1994. On complexity as bounded rationality. *Proc. 26th ACM Sympos. on Theory of Comput.*, Montreal, Quebec, Canada, 726–733.

———, ———. 1998. On bounded rationality and computation complexity. *Mimeo*, University of California, Berkeley, CA, and Bell Laboratories, Murray Hill, NJ.

Piccione, M., A. Rubinstein. 2002. Modeling economic interaction of agents with diverse abilities to recognize equilibrium patterns. *Mimeo*, Tel Aviv University, Tel Aviv, Israel, and London School of Economics, London, England, U.K.

Rubinstein, A. 1986. Finite automata play the repeated prisoners dilemma. *J. Econom. Theory* **39** 83–96.

van Lint, J. H., R. M. Wilson. 2001. *A Course in Combinatorics*, 2nd ed. Cambridge University Press, Cambridge, U.K.

O. Gossner: THEMA, UMR CNRS 7536, Université Paris X-Nanterre, 200 avenue de la République, 92001 Nanterre, France, and CORE, 34 Voie du Roman Pays, Université Catholique de Louvain, Louvain, Belgique; e-mail: ogossner@u-paris10.fr

P. Hernández: CORE, 34 Voie du Roman Pays, Université Catholique de Louvain, Louvain, Belgique; e-mail: hernandez@core.ucl.ac.be