ABILITY AND KNOWLEDGE

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Abstract. In games with incomplete information, more information to a player implies a broader strategy set for this player in the normal form game, hence more knowledge implies more ability. We prove that, on the other hand, given two normal form games G and G' such that players in a subset J of the set of players possess more strategies in G' than in G, there exist two games with incomplete information with normal forms G and G' such that players in J are more informed in the second than in the first. More ability can then be rationalized by more knowledge, and our result thus establishes the formal equivalence between ability and knowledge.

1. Introduction

"Ability" refers to the possibility of an agent to achieve particular actions. "Knowledge" refers to the information possessed by the agent. For instance, "running 100 m. in less than 12 sec." is an ability, whereas "knowing the password required to log into computer account X" refers to some knowledge. Some skills can be described either in terms of knowledge, or as abilities, as for instance "preparation of a particular recipe", or "piloting a plane". In fact, the connections between knowledge and ability are strong, and the aim of this paper is to clarify these.

Different levels of ability for a player can be represented by comparing normal form games. If an agent possesses more strategies in game G than in G', this expresses more ability for this agent in G than in G'. Knowledge is naturally represented by information structures. Given two information structures $\mathfrak{E}$ and $\mathfrak{E}'$, a player has more knowledge in $\mathfrak{E}$ than in $\mathfrak{E}'$ when his information partition is finer in $\mathfrak{E}$ than in $\mathfrak{E}'$.

An information structure together with a payoff specification with incomplete information define a game with incomplete information, that can be represented in normal form. It is well known that finer information implies larger strategy sets in the associated normal form games. Indeed, agents having more knowledge can use more information in their decision making, which results in more ability. For

Date: July 29, 2005.

instance, when a firm discovers the knowledge of some technology, this results in a larger production set.

In this paper we prove the equivalence of ability and knowledge. Since it is already well known that more knowledge implies more ability, we show a converse to this proposition, namely that more ability can always be rationalized as the consequence of more knowledge. More precisely, given two finite normal form games G and G', and assuming that players in a subset J of the set of players have more ability in G' than in G, we construct two information structures $\mathfrak{E}$ and $\mathfrak{E}'$ and a payoff specification γ, such that:

- $\mathfrak{E}'$ is more informative than $\mathfrak{E}$ for players in J,
- The normal form game associated with $\mathfrak{E}$ and γ is G
- The normal form game associated with $\mathfrak{E}'$ and γ is G'

The proof of this result relies on the following logic. Assume that in G', player i possesses a strategy a which is not available in G. We try to explain this extra strategy by extra knowledge of player i in games with incomplete information. To do this, we construct a game in which player i, in order to play strategy a, must announce a password, which is initially uniformly drawn in the continuum $[0, 1]$. If i is informed of the value of the password, i has the possibility to announce the true value whatever it is, hence to achieve a with probability 1. If i has no information of the password, the announced value will match the password with zero probability, hence a is not an available action to i. In this reasoning, the ability to play a is rationalized as the consequence of the knowledge of the adequate information. Our proof relies on a continuum space of states of the world (the passwords in our previous example). We show in section 4.3 that this assumption is needed, where we provide a counter example when this space is finite or countable. In order to relativize the importance of the assumption of an infinite set of states of the world, we also present a characterization of the reductions of strategy sets that arise from information coarsening when the space of states of the world is finite in section 4.5. This characterization allows us to understand the continuum of states of the world situation as the limit case of large but finite state spaces.

Our result demonstrates that, without imposing any further structure on the nature of knowledge of the players, the only predictable effect of an increase in information to some player is an increase of the strategy set of this player in the corresponding normal form game.

The equivalence of knowledge and information gives a better understanding of the question of value of information. It is known at least since Hirshleifer's [Hir71] work that the value of information is not always positive in economic situations, neither for the agent for receiving more information, nor for society as a whole. As pointed out by Neyman [Ney91], the reason why information can have a negative value is that other players are aware of this extra information. More information is always beneficial to the agent if other agents are ignoring it.

Some classes of games are known to show either social or private positive value of information. In decision problems (one player games), the value of information is positive if the agent is a a Bayesian expected utility maximizer. Indeed, more strategies are always beneficial, as the only choice to be made is the choice of the utility maximizing strategy. Works by Wakker [Wak88] and Chassagnon and Vergnaud [CV99] show that value of information can be negative for a non expected utility maximizer. For more than one player, the logic of socially positive value of information extends to games of common interests. Bassan, Gossner, Scarsini and Zamir [BGSZ03] show that the common interest condition is necessary and sufficient for a property of socially positive value of information to hold. The private value of information is positive in purely antagonistic zero-sum games, where finer information, or a larger strategy set, can only be beneficial to the player receiving it, and harmful for the other player. Gossner and Mertens [GM01] and Lehrer and Rosenberg [LR03a] study the value of information in these games. For general games, examples of situations with negative value of information can be found e.g. in Bassan, Scarsini and Zamir [BSZ97] or in Kamien, Tauman and Zamir [KTZ90]. Lehrer and Rosenberg [LR03b] study the maps from partitional information structures to values of games that arise as values of games with incomplete information.

Blackwell [Bla51], [Bla53] shows that a statistical experiment yields a better payoff than another in every decision problem if and only if it is more informative. Gossner [Gos00] characterizes information structures that induce more correlated equilibrium distributions than others in every game. This order between information structures is compatible with the social value of information in all games.

Our result allows to view the value of more information as the value of a larger strategy set. Of course, such a value cannot be positive in general. For instance, by deleting the "defect" strategy for both players in the prisoner's dilemma, one transforms a game with defection as unique Nash equilibrium into a game with cooperation as unique Nash outcome. Hence, more strategies for both players is harmful for them both. In other words, committing not to use some information is formally equivalent to committing not to use certain strategies, and such a commitment may have positive effects.

We introduce the comparison concepts between normal form games in section 2, and between information structures in section 3. We establish the connexion between the two in section 4, and briefly discuss applications to the value of information in section 5.

2. Normal form games

An arbitrary set I of players is fixed.

If $(X_i)_i$ is a family of sets and $J \subset I$, X_J denotes $\Pi_{i \in J} X_i$ and $\Pi_{j \neq i} X_j$ and X denotes X_I. For a family of maps $\alpha_i \colon X_i \to Y_i$, $\alpha_J \colon X \to Y$ is defined by $\alpha_J(x) = (\alpha_i(x_i))_{i \in J}$ and α denotes α_I. We use the shortcuts i for $\{i\}$, $-J$ for $I - J$. Given any set X, Id_X denotes the identity map of X.

A normal form game $G = ((S_i), g)$ is given by a strategy space S_i for each player i and by a payoff function $g \colon S \to \mathbb{R}^I$.

A *game in mixed strategies* is given by pure strategy sets $(A_i, \mathcal{A}_i)$, and by a measurable and bounded map $g \colon A \to \mathbb{R}^I$, S_i is then the set of measures on $(A_i, \mathcal{A}_i)$ and g is defined on S by $g(s) = \int g(a) ds$, where s is the product measure of $(s_i)_i$ on A.

When each A_i is finite, G is a *finite game in mixed strategies*.

Two strategies s_i, s'_i in S_i are payoff-equivalent whenever for all $s_{-i} \in S_{-i}$, $g(s_i, s_{-i}) = g(s'_i, s_{-i})$.

2.1. Equivalent games.

We now define equivalence between games.

Definition 1. *Given two normal for games G and G', G is equivalent to G', and we note $G \sim G'$, when there exists a family of mappings $\psi = (\psi_i)_i$, $\psi_i \colon S_i \to S'_i$ such that:*

1. *$g = g' \circ \psi$,*
2. *There exist maps $(e'_i)_i$, $e'_i \colon S'_i \to \mathtt{Im}\psi_i$ such that for every $K \subset I$ and $s' \in S'$, $g'(s') = g'(s'_{-K}, e'_K(s'_K))$.*

We then say that ψ is an equivalence map *from G to G'.*

Remark 1. *Condition (2) of the definition implies each strategy $s'_i \in S'_i$ is payoff equivalent to $e'_i(s'_i) \in \mathtt{Im}\psi_i$. When I is finite, this condition is equivalent to the existence for any s'_i of a payoff-equivalent strategy in $\mathtt{Im}\psi_i$. This equivalence does not hold when I is not finite, see Example 2.*

Proposition 1. *The composition of two equivalence maps is an equivalence map. In particular, $\sim$ is an equivalence relation.*

Proof. The relation $\sim$ is reflexive since the identity on G to fulfills the conditions.

We prove $\sim$ is symmetric. Assume $G \sim G'$, and let ψ, e' be the corresponding mappings. We select ψ'_i such that $\psi_i \circ \psi'_i = e'_i$, and let $e_i = \psi'_i \circ \psi_i$. Then, $g \circ \psi' = g' \circ \psi \circ \psi' = g' \circ e' = g'$. And for $K \subset I$:

$$\begin{aligned} g \circ (\mathrm{Id}_{S_{-K}}, e_K) &= g' \circ (\psi_{-K}, \psi_K \circ \psi'_K \circ \psi_K) = g' \circ (\psi_{-K}, e'_K \circ \psi_K) \\ &= g' \circ \psi = g \end{aligned}$$

To prove that $\sim$ is transitive, assume $G \sim G'$ and $G' \sim G''$, and let ψ, e', ψ', e'' be the corresponding mappings. Let $\tilde{\psi} = \psi' \circ \psi$. It is verified that $g = g'' \circ \tilde{\psi}$. Let α_i and α'_i such that $\psi_i \circ \alpha_i = e'_i$, $\psi'_i \circ \alpha'_i = e''_i$, and define $\tilde{e}''_i \colon S''_i \to \mathtt{Im}\psi_i$ by $\tilde{e}''_i = \tilde{\psi}_i \circ \alpha_i \circ \alpha'_i$. For $K \subset I$, we have:

$$\begin{aligned} g'' &= g'' \circ \psi' \circ \alpha' = g' \circ \alpha' = g' \circ (\alpha'_{-K}, e'_K \circ \alpha'_K) \\ &= g'' \circ (\psi'_{-K} \circ \alpha'_{-K}, \psi'_K \circ e'_K \circ \alpha'_K) \\ &= g''(e''_{-K}, \tilde{e}''_K) = g''(\mathrm{Id}_{S''_{-K}}, \tilde{e}''_K) \end{aligned}$$

□

Example 1. *G and G' are two finite games in mixed strategies given by the payoff matrices:*

	l	m	r
t	$1,0$	$5,2$	$3,1$
b	$5,0$	$3,6$	$4,3$

G

	L	R
T	$1,0$	$5,2$
M	$5,0$	$3,6$
B	$3,0$	$4,4$

G'

Define ψ_1 and ψ_2 on pure strategies by $\psi_1(t) = T$, $\psi_1(b) = M$, $\psi_2(l) = L$, $\psi_2(m) = R$, $\psi_2(r) = \frac{1}{2}L + \frac{1}{2}R$, and extend these maps linearly to the mixed strategy spaces. Then, $g = g' \circ \psi$, and to see that every strategy in G' is payoff equivalent to a strategy in the image of ψ, note that B is payoff equivalent to $\frac{1}{2}T + \frac{1}{2}M$.

The following example shows some difficulties that may arise with an infinite number of players.

Example 2. *The set of players is the set of integer numbers. G is given by $S_i = \{A\}$ and $g_i \equiv 0$, G' is given by $S_i = \{a, b\}$, $g_i(s) = 1$ if $\#\{i, s_i = b\} = \infty$, and $g_i(s) = 0$ otherwise. The maps $\psi_i \colon S_i \to S'_i$ given by $\psi_i(A) = a$ verify $g = g' \circ \psi$, and since a and b are payoff equivalent strategies in G', every strategy in G' is payoff-equivalent to an element of $\mathtt{Im}\psi_i$. Note that condition (2) of definition 1 is not satisfied. In fact, G and G' are not equivalent since it is impossible to construct a map ψ' from G' to G such that $g' = g \circ \psi'$.*

2.2. **Restrictions of games.** Deleting elements of the strategy space for player i transforms a game G into a game G' in which allows less strategic choices for player i. More generally, the following definition captures the fact that more strategies are available for a subset J of players in G' than in G.

Definition 2. *G is a restriction for players in J of G', and we note $G \subseteq_J G'$, when there exists a family of mappings $\varphi = (\varphi_i)_i$, $\varphi_i \colon S_i \to S'_i$ such that:*

(1) $g = g' \circ \varphi$,
(2) There exist maps $(e'_i)_{i \notin J}$, $e'_i \colon S'_i \to \mathtt{Im}\varphi_i$, $\forall s' \in \mathtt{Im}\varphi_J \times S'_{-J}$, $\forall K \subset I - J$, $g'(s') = g'(e'_K(s'_K), s'_{-K})$.

We then say that φ is a restriction map*, or J-*restriction map *from G to G'.*

Remark 2. *It follows from the definitions that $G \sim G'$ if and only if $G \subseteq_\emptyset G'$ and that $G \subseteq_J G'$ implies $G \subseteq_{J'} G'$ whenever $J \subseteq J'$.*

Remark 3. *Condition (2) of the definition implies each strategy $s'_i \in S'_i$ is payoff equivalent to $e'_i(s'_i) \in \mathtt{Im}\psi_i$. When I is finite, the two conditions are equivalent, otherwise they are not, see example 2.*

Proposition 2. *The composition of two J-restriction maps is a restriction map. In particular, the relations $\subseteq_J$ are transitive.*

Proof. Let φ and φ' be the restriction maps from G to G' and from G' to G'', and let e'_{-J}, e''_{-J} be the corresponding maps on S'_J and S''_J. Letting $\tilde{\varphi} = \psi' \circ \varphi$, we have $g = g'' \circ \tilde{\varphi}$.

For $i \notin J$ let α_i and α'_i such that $\varphi_i \circ \alpha_i = e'_i$, $\varphi'_i \circ \alpha'_i = e''_i$, and define $\tilde{e}''_i \colon S''_i \to \mathtt{Im}\varphi_i$ by $\tilde{e}''_i = \tilde{\varphi}_i \circ \alpha_i \circ \alpha'_i$. For $K \subset I - J$, we have:

$$\begin{aligned} g'' &= g'' \circ \psi' \circ \alpha' = g' \circ \alpha' = g' \circ (\alpha'_{-K}, e'_K \circ \alpha'_K) \\ &= g'' \circ (\psi'_{-K} \circ \alpha'_{-K}, \psi'_K \circ e'_K \circ \alpha'_K) \\ &= g''(e''_{-K}, \tilde{e}''_K) = g''(\mathrm{Id}''_{S_{-K}}, \tilde{e}''_K) \end{aligned}$$

□

Example 3. *Consider the finite games in mixed strategies G and G' given by the payoff matrices:*

	l	r
t	$1,-1$	$-1,1$
b	$0,0$	$0,0$

G

	L	R
T	$1,-1$	$-1,1$
B	$-1,1$	$1,-1$

G'

Define φ_1 and φ_2 on pure strategies by $\varphi(t) = T$, $\varphi(b) = \frac{1}{2}T + \frac{1}{2}B$, $\varphi(l) = L$, $\varphi(r) = R$ and extend these maps linearly to the mixed strategy spaces. Then, φ, verifies the properties of the definition with $J = \{1\}$, so $G \subseteq_1 G'$. In fact, G is a version of G' in which player 1 is restricted to play mixed strategies that put weight no more than $\frac{1}{2}$ on B.

Remark 4. *If $-J$ is finite, condition (2) of definition 2 can be replaced by: There exist maps $(e'_i)_{i \notin J}$, $e'_i \colon S'_i \to \mathtt{Im}\varphi_i$, $\forall s' \in \mathtt{Im}\varphi_J \times S'_{-J}$, $\forall i \notin J$, $g'(s') = g'(e'_i(s'_i), s'_{-i})$.*

Remark 5. *Point (2) of definition 2 imposes that for $i \notin J$, for any element of S'_i there exists an element $e'_i(s'_i)$ of $\mathtt{Im}\varphi_i$ that is payoff equivalent to s'_i against all elements of $\mathtt{Im}\varphi_J \times S'_{-J-i}$. Note that $e'_i(s'_i)$ is not necessarily payoff equivalent to s'_i, as shown by next example.*

Example 4. *Consider the finite games in pure strategies G and G' given by the payoff matrices:*

	l
t	$0,0$

G

	L	R
T	$0,0$	$1,1$
B	$0,0$	$2,2$

G'

The map φ defined by $\varphi(t, l) = (T, L)$ is a 2-restriction map from G to G'. Note however that B is not payoff-equivalent to T.

2.3. **Affine restrictions.** We prove in this section that when I is finite and G, G' are finite games in mixed strategies, the restriction maps can be taken affine.

Proposition 3. *Assume I is finite and G, G' are finite games in mixed strategies. Then there exists affine maps $(\varphi_i)_i$ such that φ is a J-restriction from G to G'.*

Proof. Let φ be a J-restriction map from G to G'. Define $\tilde{\varphi}$ by $\tilde{\varphi}_i(s_i) = \mathbf{E}_{s_i}\varphi_i(a_i)$. For $s \in S$, $g(s) = \mathbf{E}_s g(a) = \mathbf{E}_s g'(\varphi(a)) = g'(\mathbf{E}_s\varphi(a)) = g'(\tilde{\varphi}(s))$. Let e_{-J} verify condition (2) of definition 2 for φ, and for $i \notin J$ let α_i be such that $\varphi_i \circ \alpha_i = e'_i$. Let then $\tilde{e}'_i = \tilde{\varphi}_i \circ \alpha_i$. For $s'_J = \varphi_J(a_J)$ and $s'_{-J} \in S'_{-J}$, and $K \subset -J$:

$$\begin{aligned} g'(s') &= g'(\varphi_J(a_J), \varphi_{-J}(\alpha_{-J}(s'_{-J}))) = g(a_J, \alpha_{-J}(s'_{-J})) \\ &= \mathbf{E}_{\alpha_K(s'_K)} g(a_J, \alpha_{-J-K}(s'_{-J-K}), a_K) \\ &= \mathbf{E}_{\alpha_K(s'_K)} g'(\varphi(a_J), \varphi_{-J-K}(\alpha_{-J-K}(s'_{-J-K})), \varphi_K(a_K)) \\ &= g'(s'_{-K}, \tilde{e}_i(s'_K)) \end{aligned}$$

This relation extends linearly to $s'_J \in \mathtt{Im}\tilde{\varphi}_J \times S'_{-J}$. □

Remark 6. *An equivalence that is affine and onto from a finite game with finite number of players to another is a* reduction *in the sense of Mertens [Mer03]. Since there always exists a reduction from finite game in mixed strategies with finite number of players to its reduced normal form (see [VJ98]), we deduce that every such game is equivalent to its reduced normal form.*

2.4. **Restrictions and equivalences.** The aim of this section is to address the following question: Assume that G is a restriction (for any subset J of the players, or more generally for $J = I$) of G', and that G' is a restriction of G. Can we infer that G and G' are equivalent? Answering this question helps clarifying the connections between equivalences and restrictions.

We first provide a counter-example to this conjecture for general games.

Example 5. *We consider a version of an "iron arm" fight in which player's strengths may vary. There are 2 players, 1 and 2. In G, player i chooses some energy put in the fight, $a_i \in [0, 1]$. The payoff to player i is 1 is $a_i > a_{3-i}$ (i wins the fight), 0 if $a_i = a_{3-i}$ (draw), and -1 is $a_i < a_{3-i}$ (i loses the fight). The game G' is the same as G except that player 1's strategy set is $[0, 2]$. The game G'' is the same as G except that both player's strategy sets are $[0, 2]$. Considering the maps $\psi_i \colon a_i \mapsto 2a_i$ from $[0, 1]$ to $[0, 2]$ show that G and G'' are equivalent. By definition of the games, $G \subseteq_1 G'$ and $G' \subseteq_2 G''$, hence $G \subseteq_{\{1,2\}} G' \subseteq_{\{1,2\}} G$. But G and G' are not equivalent: indeed, player 1 has a strategy that guarantees a win in G', but not in G.*

The previous counter example relies on infinite pure strategy spaces. We now state a positive answer for finite games in mixed strategies.

Theorem 1. *Assume that I is finite and G and G' are finite games in mixed strategies such that $G \subseteq_I G'$ and $G' \subseteq_I G$, then $G \sim G'$.*

We start with a lemma.

Lemma 1. *If I is finite, G is a reduced normal form finite game in mixed strategies$\phi = (\phi_i)_i$ a family of maps such that $g \circ \phi = g$, then each ϕ_i is a permutation of A_i.*

Proof. Let M be the $A_i \times A_{-i}$ matrix with elements in $\mathbb{R}^I$ defined by $M_{a_i,a_{-i}} = g(a_i, a_{-i})$. Let S and T be the transition matrices over A_i and A_{-i} respectively given by $S_{a_i,b_i} = \phi_i(a_i)(b_i)$ and $T_{a_{-i},b_{-i}} = \phi_{-i}(a_{-i})(b_{-i})$. The relation $g = g \circ \phi$ rewrites $M = SM\,{}^tT$. Let $k \in \mathbb{N}$ be such that both S^k and T^k are transitions of aperiodic Markov chains, and let S^∞ and T^∞ denote the limits of the sequences $(S^{nk})_n$ and $(T^{nk})_n$. We deduce from the above that $M = S^\infty M\,{}^t T^\infty$. Define ϕ^∞ by $\phi^\infty_i(a_i)(b_i) = S^\infty_{a_i,b_i}$, $\phi^\infty_{-i}(a_{-i})(b_{-i}) = T_{a_{-i},b_{-i}}$. Since $\phi^\infty_i \circ \phi^\infty_i = \phi^\infty_i$:

$$g = g \circ \phi^\infty = g \circ (\phi^\infty_i \circ \phi^\infty_i, \phi^\infty_{-i}) = g \circ (\phi^\infty_i, \mathrm{Id}_{S_{-i}})$$

So that each a_i is payoff-equivalent to $\phi^\infty_i(a_i)$, hence $a_i = \phi^\infty_i(a_i)$, S^∞ is the identity matrix, and ϕ_i is a permutation of A_i. □

Proof of theorem 1. From remark 6, remark 2 and proposition 2, it suffices to prove the theorem when G and G' are reduced normal forms. From proposition 3, there exist linear J-restriction maps φ and φ' from G to G' and from G' to G, and let e, e' be the corresponding maps on S, S'. , and let $\phi = \varphi' \circ \varphi$. Then ϕ is an inclusion map, and by lemma 1 each ϕ_i is a permutation on A_i, thus a linear isomorphism on S_i.

We now prove that each ϕ_i defines is surjective. For $a'i_\in A'_i$, let $s_i = \phi_i^{-1}(\phi'_i(a_i))$ and $s'_i = \varphi_i(s_i)$. Then $\phi_i(a_i) = \phi'_i(s_i)$ and for every $s'_{-i} \in S'_{-i}$,

$$g'(a'_i, s'_{-i}) = g(\varphi'_i(a'_i), \varphi'_{-i}(s'_{-i})) = g(\varphi'_i(s'_i), \varphi'_{-i}(s'_{-i})) = g'(a'_i, s'_{-i})$$

so that a'_i and s'_i are payoff equivalent. Since G' is a reduced normal form $a'_i = s'_i$. Hence $a'_i \in \mathtt{Im}\varphi_i$, $\mathtt{Im}\varphi_i = S'_i$. This implies that φ is an equivalence map from G to G' (take $e' = \mathrm{Id}_{S'}$). □

3. Knowledge: Comparison of information structures

3.1. **Description of information.** K is a measurable space of states of nature. An information structure is given by $\mathfrak{E} = (\Omega, \mathcal{E}, P, (\mathcal{E}_i)_i, \kappa)$, where $(\Omega, \mathcal{E}, P)$ is a probability space of states of the world, $\mathcal{E}_i$ is a sub σ-algebra of $\mathcal{E}$ that describes the information of player i, and κ is a $\mathcal{E}$-measurable application to K that describes the state of the nature.

Definition 3. *We say that $\mathfrak{E}$ is less informative for players in J than $\mathfrak{E}'$, and we note $\mathfrak{E} \subseteq_J \mathfrak{E}'$ when $\mathfrak{E}$ can be obtained from $\mathfrak{E}'$ by replacing the σ-algebras $\mathcal{E}'_j$ by sub σ-algebras $\mathcal{E}_j$ for $j \in J$.*

Example 6. *Choose $\Omega = K = \{k_1, k_2\}$ endowed with the discrete σ-algebra and the uniform probability, and κ is the identity. Set $\mathcal{E}'_1 = \mathcal{E}_1 = \mathcal{E}_2 = \{\emptyset, \Omega\}$, and $\mathcal{E}'_2$ the discrete σ-algebra. Then, player 1 is never informed of k, whereas player 2 knows k in $\mathfrak{E}'$ but not in $\mathfrak{E}$. We have $\mathfrak{E} \subseteq_2 \mathfrak{E}'$.*

3.2. **Games of incomplete information.** For a given space of states of nature K, a payoff specification is given by measurable spaces X_i and by a measurable and bounded payoff function with incomplete information $\gamma \colon \Pi_i X_i \times K \to \mathbb{R}^I$.

An information structure $\mathfrak{E}$ and a payoff specification γ on the same space K define a normal form game $G(\mathfrak{E}, \gamma)$ in which a strategy for player i is a measurable map f_i from $\mathcal{E}_i$ to X_i and payoffs are given by the relation $g_{\mathfrak{E},\gamma}(f) = \mathbf{E}_P \gamma((f_i)(\omega), \kappa(\omega))$.

Example 7. *Take up the information structures $\mathfrak{E}$ and $\mathfrak{E}'$ of example 6, and let $X_1 = \{T, B\}$, $X_2 = \{L, R\}$, and γ be given by the two payoff matrices:*

	L	R
T	$0,0$	$1,2$
B	$2,0$	$0,2$

$k = 1$

	L	R
T	$0,1$	$1,0$
B	$2,1$	$0,0$

$k = 2$

In $G_{\mathfrak{E},\gamma}$ the only strategies for $i \in \{1, 2\}$ are the constant ones in X_i, and the payoff matrix of this game is:

	L	R
T	$0, \frac{1}{2}$	$1,1$
B	$2, \frac{1}{2}$	$0,1$

$G_{\mathfrak{E},\gamma}$

In $G_{\mathfrak{E}',\gamma}$ the strategies for player 1 are the constant ones, and player 2 has 4 strategies. For instance LR is the strategy of player 2 that plays L if $k = k_1$ and R if $k = k_2$. The payoff matrix of this game is

	LL	LR	RL	RR
T	$0, \frac{1}{2}$	$\frac{1}{2}, 0$	$\frac{1}{2}, \frac{3}{2}$	$1,1$
B	$2, \frac{1}{2}$	$1,0$	$1, \frac{1}{2}$	$0,1$

$G_{\mathfrak{E}',\gamma}$

4. Relations between knowledge and ability

4.1. More knowledge implies more ability.

We recall the well known fact that more knowledge implies more ability.

Proposition 4. *$\mathfrak{E} \subseteq_J \mathfrak{E}'$ implies $G_{\mathfrak{E},\gamma} \subseteq_J G_{\mathfrak{E}',\gamma}$.*

Proof. Let Σ_i and Σ'_i be the sets of measurable maps from $(\Omega, \mathcal{E}_i)$ and $(\Omega, \mathcal{E}'_i)$ respectively to X_i. It is straightforward that the family of inclusion maps ψ_i from Σ_i to Σ'_i verifies the conditions of definition 2. □

Example 8. *It is seen in the previous example that $G_{\mathfrak{E},\gamma} \subseteq_J G_{\mathfrak{E}',\gamma}$.*

4.2. Question about a converse theorem.

Given K, and two games such that $G \subseteq_J G'$, we address the existence of $\mathfrak{E}$, $\mathfrak{E}'$, and γ, such that

- $G_{\mathfrak{E},\gamma} \sim G$;
- $G_{\mathfrak{E}',\gamma} \sim G'$;
- $\mathfrak{E} \subseteq_J \mathfrak{E}'$.

4.3. **A counter example if Ω is finite or countable.** Let G and G' be the one-player finite games in mixed strategies:

b	0

G

T	1
B	0

G'

Proposition 5. *Consider the above games G and G', and assume Ω is finite or countable. There does not exist $\mathfrak{E}$, $\mathfrak{E}'$, and γ, such that*

- $G_{\mathfrak{E},\gamma} \sim G$;
- $G_{\mathfrak{E}',\gamma} \sim G'$;
- $\mathfrak{E} \subseteq_1 \mathfrak{E}'$.

Proof. By contradiction. By deleting elements of Ω that have null probability and merging elements which are not separated by $\mathcal{E}'$, we reduce to the case where $\mathcal{E}'$ is the discrete σ-algebra and $P(\omega) > 0$ for all ω. We also assume wlog. that $\mathcal{E} = \{\emptyset, \Omega\}$. From the equivalence between G' and $G_{\mathfrak{E}',\gamma}$, we deduce that $\min_{(x_\omega)_\omega} \sum P(\omega) g(x_\omega, \kappa(\omega))$ is well defined and equals 0. Hence, for each $\omega \in \Omega$ there exists x_ω that minimizes $g(x_\omega, \omega)$. For every $x \in X$:

$$\sum\nolimits_\omega P(\omega) g(x, \kappa(\omega)) \geq \sum\nolimits_\omega P(\omega) g(x_\omega, \kappa(\omega))$$

with strict inequality if there exists ω such that $g(x, \kappa(\omega)) > g(x_\omega, \kappa(\omega))$. By equivalence of G and $G_{\mathfrak{E},\gamma}$, $\sum_\omega P(\omega) g(x, \kappa(\omega)) = 0$ for every x. Hence, for every x, $g(x, \kappa(\omega)) = g(x_\omega, \kappa(\omega))$. Therefore $\sum_\omega P(\omega) g(x'_\omega, \kappa(\omega))$ is independent of $(x'_\omega)_\omega$, so that the payoff function of G' must be identically 0. A contradiction. □

4.4. **A positive result.**

Theorem 2. *Given any pair of games in mixed strategies such that $G \subseteq_J G'$, there exists K, $\mathfrak{E}$, $\mathfrak{E}'$ and γ, such that:*

(1) $G_{\mathfrak{E},\gamma} \sim G$;
(2) $G_{\mathfrak{E}',\gamma} \sim G'$;
(3) $\mathfrak{E} \subseteq_J \mathfrak{E}'$.

Proof. We construct the information structures and the payoff specification, and then verify the equivalences of games. Let φ, e'_{-J} be the maps from G to G' as in definition 2.

The information structures Let $(K_j, \mathcal{K}_j, \beta_j)$ for $j \in J$ be independent copies of $[0,1]$ endowed with the Borel sets and the Lebesgue measure, and let $(\Omega, \mathcal{E}, P)$ be the product of these spaces. We let $K = \Omega$ and κ be the identity map.

For every $i \in I$, $\mathcal{E}_i = \{\emptyset, \Omega\}$. For $j \in J$, $\mathcal{E}'_j$ is generated by $\mathcal{K}_i$ and for $j \notin J$ $\mathcal{E}'_j = \mathcal{E}_j$. It is thus verified that $\mathfrak{E} \subseteq_J \mathfrak{E}'$.

Payoff specification Assume wlog. that the S_i's and S'_i's are disjoint. For $i \notin J$ let $X_i = S_i$ and for $i \in J$ let $X_i = (S_i \cup S'_i) \times K_i$. We endow X_i with the product of the power class $2^{S_i \cup S'_i}$ and $\mathcal{K}_i$. For $i \in J$ we define an outcome function o_i from $X_i \times K_i$ to $(S_i \cup S'_i, 2^{S_i \cup S'_i})$: select $s_i^0 \in S_i$, and let

$$\begin{cases} o_i((s_i, b_i), k_i) = s_i & \text{if } s_i \in S_i \\ o_i((s'_i, b_i), k_i) = s'_i & \text{if } s'_i \in S'_i \text{ and } b_i = k_i \\ o_i((s'_i, b_i), k_i) = s_i^0 & \text{if } s'_i \in S'_i \text{ and } b_i \neq k_i \end{cases}$$

For $C \subset S_i \cup S'_i$, $o_i^{-1}(C) = \{s_i \in S_i \cap C\} \cup \{b_i = k_i, s_i \in S'_i \cap C\}$ if $s_i^0 \notin C$ and $o_i^{-1}(C) = \{s_i \in S_i \cap C\} \cup \{b_i = k_i, s_i \in S'_i \cap C\} \cup \{b_i \neq k_i, s_i \in S'_i\}$ if $s_i^0 \in C$, hence it is a measurable event. So o_i is measurable.

For $i \in J$ define $\tilde{o}_i$ from $X_i \times K_i$ to $(S'_i, 2^{S'_i})$ by $\tilde{o}_i(x_i, k_i) = o_i(x_i, k_i)$ if $o_i(x_i, k_i) \in S'_i$ and $\tilde{o}_i(x_i, k_i) = \varphi_i(o_i(x_i, k_i))$ if $o_i(x_i, k_i) \in S_i$. For $i \notin J$, let $\tilde{o}_i = \varphi_i$. This defines a measurable map $\tilde{o} = (\tilde{o}_i)_i \colon X \times K \to S'$. The payoff function with incomplete information is $\gamma = g' \circ \tilde{o}$. Note that g' is measurable from the product of the sets $(S'_i, 2^{S'_i})$, hence γ is measurable.

Verification of (2) For $i \notin J$, any strategy f'_i in $G_{\mathfrak{E}',\gamma}$ plays constantly some $s_i \in S_i$, and we let $\psi'_i(f'_i) = \tilde{o}_i(s_i)$. For $i \in J$. Given any strategy $f'_i \colon (\Omega, \mathcal{E}'_i) \to X_i$ and $C \in \mathcal{A}'_i$, the map $\omega \mapsto \tilde{o}_i(f'_i(\omega), k_i(\omega))(C)$ is measurable as the composition of measurable maps, and we define:

$$\psi'_i(f'_i)(C) = \int_\Omega \tilde{o}_i(f'_i(\omega), k_i(\omega))(C) dP$$

From the monotone convergence theorem, $\psi'_i(f'_i)$ is σ-additive, hence is a probability measure on $(A'_i, \mathcal{A}'_i)$. Given a profile f', $\psi'(f')$ denotes the product probability measure of $(\psi'_i(f'_i))_i$.

For any $C \in \Pi_i 2^{S_i}$, $\int_\Omega \mathbb{I}_C \tilde{o}(f'(\omega), k) dP = \int_\Omega \mathbb{I}_C d\psi'(f')$, hence for the $\Pi_i 2^{S_i}$-measurable map g', $\int_\Omega g' \tilde{o}(f'(\omega), k) dP = \int_\Omega g d\psi'(f')$, which implies $g_{\mathfrak{E}',\gamma}(f') = g'(\psi'(f'))$. This establishes point (1) of definition 1 for G'. We now check point (2) of this definition: If $j \in J$, $\mathtt{Im}\psi'_i = S'_i$ for every $s'_i \in S'_i$, f'_i given by $f'_i(\omega) = (s'_i, k_i)$ is such that $\psi'_i(f'_i) = s'_i$, so we set $\tilde{e}'_i = \mathrm{Id}_{S'_i}$. For $j \notin J$, $\mathtt{Im}\psi'_i \supset \mathtt{Im}\varphi_i$ and we set $\tilde{e}'_i = \tilde{e}_i$. Hence for $K \subset I$, $g' = g' \circ (\mathrm{Id}_{S'_{-K \cup J}}, e'_{-J \cap K}) = g' \circ (\mathrm{Id}_{S'_{-K}}, \tilde{e}'_K)$.

Verification of (1) For $i \notin J$, any strategy f_i in $G_{\mathfrak{E}',\gamma}$ plays constantly some $s_i \in S_i$, and we let $\psi_i(f_i) = s_i$. For $i \in J$, any strategy f_i in $G_{\mathfrak{E}',\gamma}$ is such that $o_i(f_i(\omega), k_i)$ equals P almost surely some $s_i \in S_i$, and we let $\psi_i(f_i) = s_i$. In both cases $\psi'_i = \varphi_i \circ \psi_i$, hence

$g_{\mathfrak{E},\gamma}(f) = g' \circ \psi'(f) = g' \circ \varphi \circ \psi(f) = g \circ \psi(f)$, hence (1) of definition 1. For point (2), it suffices to observe that any strategy f_i that plays constantly s_i verifies $\psi_i(f_i) = s_i$, so that $\texttt{Im}\psi_i = S_i$. □

Remark that the constructed information structures $\mathfrak{E}$ and $\mathfrak{E}'$ depend on J, but not on the games G and G'. Note also that the payoff specification γ has the same image as g'. In particular, γ is zero-sum whenever g' is, and a group of players have common interests in γ whenever they do in g'. This leads us the the following statement that strengthens theorem 2.

Theorem 3. *For every subset J of players, there exist information structures $\mathfrak{E} \subseteq_J \mathfrak{E}'$ such that for any pair of games in mixed strategies such that $G \subseteq_J G'$, there exists a payoff specification γ that verifies:*
(1) $G_{\mathfrak{E},\gamma} \sim G$;
(2) $G_{\mathfrak{E}',\gamma} \sim G'$;
(3) $\texttt{Im}\gamma = \texttt{Im}g'$.

4.5. **A characterization with finitely many states of nature.** When Ω is finite, more knowledge implies more ability (see proposition 4), but the converse may fail (see proposition 5). In this section, we present a relation between games that strengthens the relation "is a restriction of" and which is equivalent to a coarsening of information when Ω is finite.

Definition 4. *Given $\varepsilon \geq 0$, G is an ε-restriction for players in J of G', and we note $G \subseteq_J^{\varepsilon} G'$, when there exists a family of mappings $\varphi = (\varphi_i)_i$, $\varphi_i \colon S_i \to S'_i$ such that:*
(1) $g = g' \circ \varphi$,
(2) For $i \notin J$, every element of S'_i is payoff equivalent to an element of $\texttt{Im}\varphi_i$,
(3) For $i \in J$ and $s'_i \in S'_i$, there exists a linear combination of elements of $\texttt{Im}\varphi_i$ which is payoff equivalent to a linear combination of elements of S'_i in which s'_i has weight no less than ε.

Remark 7. *When G is a game in mixed strategies, point 3 of the definition can be replaced by "there exists an element of $\texttt{Im}\varphi_i$ which is payoff equivalent to a linear combination of elements of S'_i in which s'_i has weight no less than ε".*

Remark 8. *When G and G' are games in mixed strategies, this point can also be replaced by "there exists an element of $\texttt{Im}\varphi_i$ which is payoff equivalent to a linear combination of elements of S'_i in which s'_i has weight ε".*

Remark 9. *Note finally that $G \subseteq_J^\varepsilon G'$ implies $G \subseteq_J^{\varepsilon'} G'$ for $\varepsilon' < \varepsilon$ and that $G \subseteq_J^0 G'$ if and only if $G \subseteq_J G'$.*

Example 9. *Consider the games G and G' of Example 3. Since b in G is payoff equivalent to $\frac{1}{2}T + \frac{1}{2}B$ in G', G is a $\frac{1}{2}$-restriction of G' for player 1.*

Example 10. *Consider the games G and G' of section 4.3. Since the strategy T in G' cannot appear with a positive weight in any linear combination of strategies of G' which is payoff equivalent to b, there exists no $\varepsilon > 0$ for which is G a ε-restriction of G' for player 1.*

The two following examples show that the games G and G' of section 4.3 are such that G can be approximated by games that are ε-restrictions of G', and G is an ε-restriction of games that are close to G', for $\varepsilon > 0$.

Example 11. *Consider the one-player finite games in mixed strategies G^ϵ for $\frac{1}{2} > \varepsilon > 0$ and G':*

$$\underset{G}{b \; \boxed{\varepsilon}} \qquad\qquad \underset{G'}{\begin{array}{c|c|} \cline{2-2} T & 1 \\ \cline{2-2} B & 0 \\ \cline{2-2} \end{array}}$$

G is an ε-restriction of G' since b is payoff equivalent to the convex combination $\varepsilon T + (1-\varepsilon)B$.

Example 12. *Consider the one-player finite games in mixed strategies G and G^ε for $1 > \varepsilon > 0$:*

$$\underset{G}{b \; \boxed{0}} \qquad\qquad \underset{G'}{\begin{array}{c|c|} \cline{2-2} T & 1 \\ \cline{2-2} B & -\varepsilon \\ \cline{2-2} \end{array}}$$

Here again, G is an $\frac{\varepsilon}{1+\varepsilon}$-restriction of G' since b is payoff equivalent to the convex combination $\frac{\varepsilon}{1+\varepsilon}T + \frac{1}{1+\varepsilon}B$.

We now state an equivalent of theorem 3 when Ω is finite.

Theorem 4. *For every finite subset J of players and $\varepsilon > 0$, there exist information structures $\mathfrak{E} \subseteq_J \mathfrak{E}'$ over a finite space Ω such that for any pair of games in mixed strategies $G \subseteq_J^\varepsilon G'$, there exists a payoff specification γ that verifies:*

(1) $G_{\mathfrak{E},\gamma} \sim G$;
(2) $G_{\mathfrak{E}',\gamma} \sim G'$;
(3) $\mathtt{Im}\gamma = \mathtt{Im}g'$.

Proof. We amend the proof of theorem 3.

The spaces $(K_j, \mathcal{K}_j, \beta_j)$ for $j \in J$ are now independent copies of $\{1, \ldots, n\}$ endowed with the discrete σ-algebras and the uniform probability measures, where $n > \frac{1}{\varepsilon}$. The construction of the information structures is otherwise unchanged.

From remark 8, we can select for each $s'_i \in S'_i$ a strategy $s_i'^0(s'_i) \in S'_i$ and a strategy such $s_i(s'_i) \in S_i$ such that $(1 - \frac{1}{n})s_i'^0(s'_i) + \frac{1}{n}s'_i$ is payoff equivalent to $\varphi_i(s_i(s'_i))$.

The spaces X_i are defined as in the proof of theorem 3. We define the outcomes functions $o_i \colon X_i \times K_i \to S_i \cup S'_i$ for $i \in J$ by:

$$\begin{cases} o_i((s_i, b_i), k_i) = s_i & \text{if } s_i \in S_i \\ o_i((s'_i, b_i), k_i) = s'_i & \text{if } s'_i \in S'_i \text{ and } b_i = k_i \\ o_i((s'_i, b_i), k_i) = s_i'^0(s'_i) & \text{if } s'_i \in S'_i \text{ and } b_i \neq k_i \end{cases}$$

The map $\tilde{o} \colon X \times K \to S'$ is defined from o as before, and the payoff function with incomplete information is $\gamma = g' \circ \tilde{o}$.

All remaining points of the proof are the same as in the proof of theorem 3, except that of $G_{\mathfrak{E},\gamma} \sim G$: For $i \notin J$ and a strategy f_i that plays constantly s_i, we let $\psi_i(f_i) = s_i$. For $i \in J$ and a strategy f_i, we let:

$$\psi_i(f_i) = \begin{cases} s_i & \text{if } f_i \text{ plays constantly } (s_i, b_i) \in S_i \times K_i \\ s_i(s'_i) & \text{if } f_i \text{ plays constantly } (s'_i, b_i) \in S'_i \times K_i \end{cases}$$

The distribution induced over A'_i by a strategy f_i is:

$$\mathbf{E}_P \tilde{o}_i(f_i(\omega), k_i) = \begin{cases} \varphi_i(s_i) & \text{if } f_i \text{ plays constantly } (s_i, b_i) \in S_i \times K_i \\ (1 - \frac{1}{n})s_i'^0(s'_i) + \frac{1}{n}s'_i & \text{if } f_i \text{ plays constantly } (s'_i, b_i) \in S'_i \times K_i \end{cases}$$

In both cases, this mixed strategy induced is payoff equivalent to $\varphi_i(\psi_i(f_i))$. From this we deduce:

$$g_{\mathfrak{E},\gamma}(f) = \mathbf{E}_P g \circ \tilde{o}(f(\omega), k) = g'(\phi(\psi(f))) = g(\psi(f))$$

Finally, we see as in the previous proof that any strategy in S_i is payoff equivalent to an element of $\texttt{Im}\psi_i$. □

5. On the value of information

More information is beneficial in one player games, socially beneficial in games with common interest, and privately beneficial for the player receiving it in zero-sum games. These results can be seen as a consequence that a broader strategy set is beneficial in these classes of games. On the other hand, many situations are known in which more information to some player may hurt this player, or the group of players. Theorem 2 can be used to construct such games with negative value of information.

Example 13. *Consider the games G and G' given by the payoff matrices:*

	L
T	$3,3$
M	$2,0$

G

	l	r
t	$3,3$	$0,4$
b	$2,0$	$1,1$

G'

Both games are dominance solvable, with $(3,3)$ as unique Nash payoff in G, and $(1,1)$ as unique Nash payoff in G'. Since G is a restriction for player 2 of G', G is equivalent to some $G_{\mathfrak{E},\gamma}$, and G' to some $G_{\mathfrak{E},\gamma}$, with $\mathfrak{E} \subseteq_2 \mathfrak{E}'$. We are then facing a situation where the value of information is negative, since the better information of player 2 in $\mathfrak{E}'$ has a negative effect on the Nash payoff for both players.

Along the same lines, it is possible to construct examples in which the value of more information for player 1 is for instance positive for player 2, but negative for player 1.

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