Cooperation and communication dynamics

Olivier Gossner

PSE

Roadmap

Games with (very) patient players

The long-run payoff

$$\gamma_\infty = \lim_{n \to \infty} \gamma_n$$

What limit do we use?

- The short answer is that in all circumstances of interest to us, the limit is always well defined.
- The technical answer is that, by using a type of limit called a Banach limit, the limit is always well-defined.

Infinitely repeated game, patient players

G_∞ is the game with payoff function γ_∞.We are interested in

1. the set E_∞ of NE payoffs of G_∞,
2. the set E'_∞ of SPNE payoffs of G_∞.

Prisoner's dilemma

	C	D
C	$3, 3$	$-1, 4$
D	$4, -1$	$0, 0$

Are there elements of E_∞ that are not $(0, 0)$?
What can be said about the elements of E_∞?

Feasible, individually rational payoffs

Recall that i can defend x^i if for every s^{-i}, there exists s^i s.t.

$$g^i(s^{-i}, s^i) \geq x^i.$$

Defending

If player i can defend x^i, and y is a NE payoff in the repeated game, then $y^i \geq x^i$.

- Let f be a profile of strategies in the repeated game (possibly behavioral).
- Let d^i be the strategy of i in the repeated game that, after history h_t, plays some $f^i(h_t)$ s.t.

$$g^i(f^i(h_t), f^{-i}(h_t)) \geq x^i$$

- After any history h_t, the expected payoff to player i at stage $t+1$ is at least x^i.
- Hence, in the repeated game, $\gamma^i(d^i, f^{-i}) \geq x^i$ (also, $\gamma^i_n(d^i, f^{-i})$).
- If f is a NE, $\gamma^i(f) \geq x^i$

Feasible, individually rational payoffs

Individually rational payoffs

- The maximum payoff i can defend is

$$\underline{v}^i = \min_{s^{-i}} \max_{s^i} g^i(s^{-i}, s^i)$$

called the min max payoff, or individually rational payoff.

- A (vector) payoff x is individually rational if for every i, $x^i \geq \underline{v}^i$.
- IR represents the set of individually rational payoffs.

Feasible payoffs

$$F = \text{co } g(A)$$

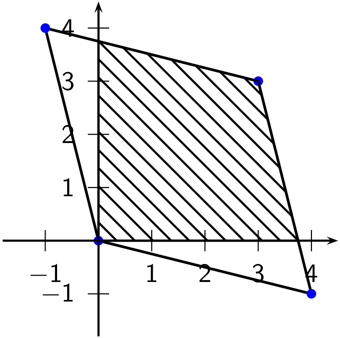
is the set of feasible payoffs.

Necessary conditions on equilibrium payoffs

Theorem

- $E_\infty \subseteq F \bigcap IR$
- For every n, $E_n \subseteq F \bigcap IR$

For the prisoner's dilemma.:



Roadmap

"The" Folk Theorem

Folk Theorem (Nash version)

$$E_\infty = F \bigcap IR$$

Let $x \in F \bigcap IR$.

- For every i, let $m_i^{-i} \in S^{-i}$ be such that

$$\max_{s^i} g^i(s^i, m_i^{-i}) = \min_{s^{-i}} \max_{s^i} g^i(s^i, s^{-i}) = v^i$$

- Consider the following strategies f:
  - Play a sequence of actions $(a_t)_t$ such that

$$\lim_n \frac{1}{n} \sum_{t=1}^{n} g(a_t) = x$$

  - If some player i "deviates", other players play m_i^{-i} forever.
- Playing f^i against f^{-i} gives x^i to i
- Playing any strategy that "deviates" at some stage gives $v^i \leq x^i$.

Example

	H	U
G	$1, 1$	$-1, -1$
S	$2, 0$	$-1, -1$

- In the game above, what is $F \bigcap IR$?
- Is $(1, 1)$ a NE payoff of the repeated game?
- What do the strategies of the proof of the Folk Theorem recommend?
- Is $(1, 1)$ a SPNE payoff of the repeated game?

Folk Theorem, perfect version

Folk Theorem (Perfect version)

$$E'_\infty = F \bigcap IR$$

Let $x \in F \bigcap IR$, and let $(a_t)_t$ such that the limit average payoff is x. Consider the following strategies f:

MP Play the sequence of action profiles $(a_t)_t$

P(i) If some player i "deviates" at stage t, other players play m_i^{-i} for t stages. After this, return to $(a_t)_t$ where it was left.

After any history:

- A strategy that deviates a finite number of times gives x^i to i.
- A strategy that deviates an infinite number of times yields
  - Deviation stages, with limit frequency 0
  - The sequence $(a_t)_t$
  - Punishment stages.
- The long-run average payoff is an average between the payoff from (a_t) and from punishment stages.

Roadmap

Preference for the present

Discount factor

1 tomorrow is equivalent to δ today, where $0 < \delta < 1$.

- δ can represent the "preference for the present" of the agents
- The game has pba. $1 - \delta$ of stopping between any two stages.
- δ-discounted payoff

$$\gamma_\delta(a_1, a_2, \ldots) = (1-\delta)\sum_{t=1}^{\infty} \delta^{t-1} g(a_t)$$

- G_δ discounted game, E'_δ set of SPNE payoffs.

Useful decomposition

$$\gamma_\delta(a_1, a_2, \ldots) = (1-\delta) g(a_1) + \delta \gamma_\delta(a_2, a_3, \ldots)$$

γ_δ is a convex combination of present payoffs and future payoffs, with weight $(1-\delta)$ on the present, and δ on the future.

Example: Prisoner's dilemma

	C	D
C	$3, 3$	$-1, 4$
D	$4, -1$	$0, 0$

For what values of δs is $(3, 3)$ a SPNE payoff of G_δ?
It is useful to think in terms of the set E'_δ. Consider a SPNE with payoff $(3, 3)$. Let x^1 be the payoff to player 1 in the subgame following (D, C).
In the subgame following (C, C), player 1 gets 3.
A necessary condition for SPNE is

$$(1-\delta)3 + \delta 3 \geq (1-\delta)4 + \delta x^1$$

What is the lowest possible value of x^1?
The necessary condition becomes $3 \geq (1-\delta)4$, or $\delta \geq 1/4$.
Can we construct a SPNE with payoff $(3, 3)$ for $\delta \geq 1/4$?

Generalization: Games with continuation payoffs

Let x be a SPNE payoff, with strategies f. Let $a^* = f(\emptyset)$. For every action profile a, let $c(a)$ be the continuation payoff in the subgame following a. A necessary condition for f to be a SPNE is that for all i, a'^i,

$$(1-\delta)g(a^{*,-i}, a'^i) + \delta c(a^{*,-i}, a'^i) \geq (1-\delta)g(a^*) + \delta c(a^*) = x^i$$

Then, x is a NE of the game with action sets A^i and payoff function

$$\pi_c(a) = (1-\delta)g(a) + \delta c(a)$$

For $E \subseteq R^I$, let $\Pi(E)$ be the union over all $c\colon a \to E$ of the NE payoffs of π_c.

We have shown that E'_δ is self-generating

$$E'_\delta \subseteq \Pi(E'_\delta)$$

We now prove $\Pi(E'_\delta) \subseteq E'_\delta$

Let $x \in \Pi(E'_\delta)$. There exists a^* and $c\colon A \to E'_\delta$ such that a^* is a NE of π_c with payoff x.
For $a \in A$, let $f(a)$ be a SPNE of G_δ with payoff $c(a)$. Consider the strategies:

- $t = 1$ Play a^* in the first stage,
- $t > 1$ Following a in the first stage, play the strategy profile $f(a)$

No deviation is profitable, either at $t = 1$ or after, these strategies form a SPNE with payoff x.

Theorem

E'_δ is a fixed point of Π:

$$E'_\delta = \Pi(E'_\delta)$$

Is it the only fixed point? Consider the prisoner's dilemma.

Fixed point characterization of E'_δ

Let E be bounded and a fixed point of Π. For $x \in E$, let $c(x) \colon E \to E$ and $a(x)$ such that $a(x)$ is a NE of $\pi_{c(x)}$ with payoff x. The strategies:

$t = 1$ Play $a(x)$, let $x_1 = c(x)(a_1)$

$t = 2$ Play $a(x_1)$, let $x_2 = c(x_1)(a_2)$

$t = 3$ ...

1) form a SPNE of G_δ, 2) with payoff x.
Finally, the union of self-generating sets, is self-generating, it is the largest fixed point.

Theorem

E'_δ is the largest bounded fixed point of Π.

Folk Theorem for discounted games

We say that a set E in R^I is full dimensional if there exists $x \in R^I$ and $\varepsilon > 0$ such that the ball of radius ε centered at x is in E. A payoff is strictly individually rational if it provides each player strictly more than the min max payoff.

Using the recursive techniques, the following can be proven.

Folk Theorem (Fudenberg Maskin 1988)

Assume that $F \bigcap IR$ is full dimensional, then for every x that is feasible and strictly individually rational, there exists δ_0 such that, for every $\delta > \delta_0$, $x \in E'_\delta$.

Conclusion

- Repetition can lead to cooperation if the game is infinitely repeated,
- Repetition does not necessarily lead to cooperation
- For infinitely repeated games with infinitely or sufficiently patient players, the set of (SP)NE payoffs is characterized by the Folk Theorem